

# “Dimension theory of arbitrary modules over finite von Neumann algebras and applications to $L^2$ -Betti numbers”

by  
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The paper is dedicated to Martin Kneser on the occasion of his seventieth birthday

## Abstract

We define for arbitrary modules over a finite von Neumann algebra  $\mathcal{A}$  a dimension taking values in  $[0, \infty]$  which extends the classical notion of von Neumann dimension for finitely generated projective  $\mathcal{A}$ -modules and inherits all its useful properties such as Additivity, Cofinality and Continuity. This allows to define  $L^2$ -Betti numbers for arbitrary topological spaces with an action of a discrete group  $\Gamma$  extending the well-known definition for regular coverings of compact manifolds. We show for an amenable group  $\Gamma$  that the  $p$ -th  $L^2$ -Betti number depends only on the  $\mathbb{C}\Gamma$ -module given by the  $p$ -th singular homology. Using the generalized dimension function we detect elements in  $G_0(\mathbb{C}\Gamma)$ , provided that  $\Gamma$  is amenable. We investigate the class of groups for which the zero-th and first  $L^2$ -Betti numbers resp. all  $L^2$ -Betti numbers vanish. We study  $L^2$ -Euler characteristics and introduce for a discrete group  $\Gamma$  its Burnside group extending the classical notions of Burnside ring and Burnside ring congruences for finite  $\Gamma$ .

**Key words:** Dimension functions for finite von Neumann algebras,  $L^2$ -Betti numbers, amenable groups, Grothendieck groups, Burnside groups.

AMS-classification numbers: 55T99, 46L99

## Introduction

Let us recall the original definition of  $L^2$ -Betti numbers by Atiyah [2]. Let  $\overline{M} \longrightarrow M$  be a regular covering of a closed Riemannian manifold  $M$  with  $\Gamma$  as group of deck transformations. We lift the Riemannian metric to a  $\Gamma$ -invariant Riemannian metric on  $\overline{M}$ . Let  $L^2\Omega^p(\overline{M})$  be the Hilbert space completion of the space  $C_0^\infty\Omega^p(\overline{M})$  of smooth  $\mathbb{R}$ -valued  $p$ -forms on  $\overline{M}$  with compact support and the standard  $L^2$ -pre-Hilbert structure. The Laplace operator  $\Delta_p$  is essentially selfadjoint in  $L^2\Omega^p(\overline{M})$ . Let  $\Delta_p = \int \lambda dE_\lambda^p$  be the spectral decomposition with right-continuous spectral family  $\{E_\lambda^p \mid \lambda \in \mathbb{R}\}$ . Let  $E_\lambda^p(\overline{x}, \overline{y})$  be the Schwartz kernel of  $E_\lambda^p$ . Since  $E_\lambda^p(\overline{x}, \overline{x})$  is an endomorphism of a finite-dimensional real vector space, its trace  $\mathrm{tr}_\mathbb{R}(E_\lambda^p(\overline{x}, \overline{x})) \in \mathbb{R}$  is defined. Let  $\mathcal{F}$  be a fundamental domain for the  $\Gamma$ -action on  $\overline{M}$ . Define the *analytic  $L^2$ -Betti number* by

$$b_p^{(2)}(\overline{M}) := \int_{\mathcal{F}} \mathrm{tr}_\mathbb{R}(E_0^p(\overline{x}, \overline{x})) d\mathrm{vol}_{\overline{x}} \in [0, \infty). \quad (0.1)$$

By means of a Laplace transformation this can also be expressed in terms of the heat kernel  $e^{-t\Delta_p}(\overline{x}, \overline{y})$  on  $\overline{M}$  by

$$b_p^{(2)}(\overline{M}) = \lim_{t \rightarrow \infty} \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{R}}(e^{-t\Delta_p}(\overline{x}, \overline{x})) d\mathrm{vol}_{\overline{x}} \in [0, \infty). \quad (0.2)$$

The  $p$ -th  $L^2$ -Betti number measures the size of the space of smooth harmonic  $L^2$ -integrable  $p$ -forms on  $\overline{M}$  and vanishes precisely if there is no such non-trivial form. For a survey on  $L^2$ -Betti numbers and related invariants like Novikov-Shubin invariants and  $L^2$ -torsion and their applications and relations to geometry, spectral theory, group theory and  $K$ -theory we refer for instance to [22, section 8], [28], [35] and [41]. In this paper, however, we will not deal with the analytic side, but take an algebraic point of view.

The  $L^2$ -Betti numbers can also be defined in an algebraic manner. Farber [20],[21] has shown that the category of finitely generated Hilbert  $\mathcal{A}$ -modules for a finite von Neumann algebra  $\mathcal{A}$  can be embedded in an appropriate abelian category and that one can treat  $L^2$ -homology from a homological algebraic point of view. Farber gives as an application for instance an improvement of the Morse inequalities of Novikov and Shubin [39], [40] in terms of  $L^2$ -Betti numbers by taking the minimal number of generators into account. An equivalent more algebra oriented approach is developed in [34] where it is shown that the category of finitely generated projective modules over  $\mathcal{A}$ , viewed just as a ring, is equivalent to the category of finitely generated Hilbert  $\mathcal{A}$ -modules and that the category of finitely presented  $\mathcal{A}$ -modules is an abelian category. This allows to define for a finitely generated projective  $\mathcal{A}$ -module  $P$  its *von Neumann dimension*

$$\dim_{\mathcal{A}}(P) \in [0, \infty) \quad (0.3)$$

by using the classical definition for finitely generated Hilbert  $\mathcal{A}$ -modules in terms of the von Neumann trace of a projector. This will be reviewed in Section 1.

In Section 2 we will prove the main technical result of this paper that this dimension can be extended to arbitrary  $\mathcal{A}$ -modules if one allows that the value may be infinite (what fortunately does not happen in a lot of interesting situations). Moreover, this extension inherits all good properties from the original definition for finitely generated Hilbert  $\mathcal{A}$ -modules such as Additivity, Cofinality and Continuity and is uniquely determined by these properties. More precisely, we will introduce

**Definition 0.4** *Define for a  $\mathcal{A}$ -module  $M$*

$$\dim'(M) := \sup\{\dim(P) \mid P \subset M \text{ finitely generated projective } \mathcal{A}\text{-submodule}\} \in [0, \infty].$$

■

Recall that the *dual module*  $M^*$  of a left  $\mathcal{A}$ -module is the left  $\mathcal{A}$ -module  $\mathrm{hom}_{\mathcal{A}}(M, \mathcal{A})$  where the  $\mathcal{A}$ -multiplication is given by  $(af)(x) = f(x)a^*$  for  $f \in M^*$ ,  $x \in M$  and  $a \in \mathcal{A}$ .

**Definition 0.5** *Let  $K$  be a  $\mathcal{A}$ -submodule of the  $\mathcal{A}$ -module  $M$ . Define the closure of  $K$  in  $M$  to be the  $\mathcal{A}$ -submodule of  $M$*

$$\overline{K} := \{x \in M \mid f(x) = 0 \text{ for all } f \in M^* \text{ with } K \subset \ker(f)\}.$$

For a finitely generated  $\mathcal{A}$ -module  $M$  define the  $\mathcal{A}$ -submodule  $\mathbf{T}M$  and the  $\mathcal{A}$ -quotient module  $\mathbf{P}M$  by:

$$\begin{aligned}\mathbf{T}M &:= \{x \in M \mid f(x) = 0 \text{ for all } f \in M^*\}; \\ \mathbf{P}M &:= M/\mathbf{T}M. \quad \blacksquare\end{aligned}$$

The notion of  $\mathbf{T}M$  and  $\mathbf{P}M$  corresponds in [20] to the torsion part and the projective part. Notice that  $\mathbf{T}M$  is the closure of the trivial submodule in  $M$ . It is the same as the kernel of the canonical map  $i(M) : M \longrightarrow (M^*)^*$  which sends  $x \in M$  to the map  $M^* \longrightarrow \mathcal{A} \quad f \mapsto f(x)^*$ . We will prove for a finite von Neumann algebra  $\mathcal{A}$  in Section 2

**Theorem 0.6** 1.  $\mathcal{A}$  is semi-hereditary, i.e. any finitely generated submodule of a projective module is projective;

2. If  $K \subset M$  is a submodule of the finitely generated  $\mathcal{A}$ -module  $M$ , then  $M/\overline{K}$  is finitely generated and projective and  $\overline{K}$  is a direct summand in  $M$ ;

3. If  $M$  is a finitely generated  $\mathcal{A}$ -module, then  $\mathbf{P}M$  is finitely generated projective and

$$M \cong \mathbf{P}M \oplus \mathbf{T}M;$$

4. The dimension  $\dim'$  has the following properties:

(a) *Continuity*

If  $K \subset M$  is a submodule of the finitely generated  $\mathcal{A}$ -module  $M$ , then:

$$\dim'(K) = \dim'(\overline{K});$$

(b) *Cofinality*

Let  $\{M_i \mid i \in I\}$  be a cofinal system of submodules of  $M$ , i.e.  $M = \cup_{i \in I} M_i$  and for two indices  $i$  and  $j$  there is an index  $k$  in  $I$  satisfying  $M_i, M_j \subset M_k$ . Then:

$$\dim'(M) = \sup\{\dim'(M_i) \mid i \in I\};$$

(c) *Additivity*

If  $0 \longrightarrow M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \longrightarrow 0$  is an exact sequence of  $\mathcal{A}$ -modules, then:

$$\dim'(M_1) = \dim'(M_0) + \dim'(M_2),$$

where  $r + s$  for  $r, s \in [0, \infty]$  is the ordinary sum of two real numbers if both  $r$  and  $s$  are not  $\infty$  and is  $\infty$  otherwise;

(d) *Extension Property*

If  $M$  is finitely generated projective, then:

$$\dim'(M) = \dim(M);$$

(e) If  $M$  is a finitely generated  $\mathcal{A}$ -module, then:

$$\begin{aligned}\dim'(M) &= \dim(\mathbf{P}M); \\ \dim'(\mathbf{T}M) &= 0;\end{aligned}$$

(f) The dimension  $\dim'$  is uniquely determined by Continuity, Cofinality, Additivity and the Extension Property. ■

In the sequel we write  $\dim$  instead of  $\dim'$ . In Section 3 we will show for an inclusion  $i : \Delta \longrightarrow \Gamma$  that the dimension function is compatible with induction with the induced ring homomorphism  $i : \mathcal{N}(\Delta) \longrightarrow \mathcal{N}(\Gamma)$  and that  $\mathcal{N}(\Gamma)$  is faithfully flat over  $\mathcal{N}(\Delta)$ . (Theorem 3.3). This is important if one wants to relate the  $L^2$ -Betti numbers of a regular covering to the ones of the universal covering. We will prove that  $\Gamma$  is non-amenable if and only if  $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}$  is trivial (Lemma 3.4.2). This generalizes the result of Brooks [7] (Remark 4.11).

In Section 4 we use this generalized dimension function to define for a (discrete) group  $\Gamma$  and a  $\Gamma$ -space  $X$  its  $p$ -th  $L^2$ -Betti number by

$$b_p^{(2)}(X; \mathcal{N}(\Gamma)) := \dim_{\mathcal{N}(\Gamma)} (H_p^\Gamma(X; \mathcal{N}(\Gamma))) \in [0, \infty], \quad (0.7)$$

where  $H_p^\Gamma(X; \mathcal{N}(\Gamma))$  denotes the  $\mathcal{N}(\Gamma)$ -module given by the singular homology of  $X$  with coefficients in the  $\mathcal{N}(\Gamma)$ - $\mathbb{Z}\Gamma$ -bimodule  $\mathcal{N}(\Gamma)$  (Definition 4.1). This definition agrees with Atiyah's definition 0.1 if  $X$  is the total space and  $\Gamma$  the group of deck transformations of a regular covering of a closed Riemannian manifold. We will compare our definition also with the one of Cheeger and Gromov [11, section 2] (Remark 4.12). In particular we can define for an arbitrary (discrete) group  $\Gamma$  its  $p$ -th  $L^2$ -Betti number

$$b_p^{(2)}(\Gamma) := b_p^{(2)}(E\Gamma; \mathcal{N}(\Gamma)) \in [0, \infty], \quad (0.8)$$

where  $E\Gamma \longrightarrow B\Gamma$  is the universal  $\Gamma$ -principal bundle. These generalizations inherit all the useful properties from the original versions and it pays off to have them at hand in this generality. For instance if one is only interested in the  $L^2$ -Betti numbers of a group  $\Gamma$  for which  $B\Gamma$  is a  $CW$ -complex of finite type and hence the original (simplicial) definition does apply, it is important to have the more general definition available because such a group  $\Gamma$  may contain an interesting normal subgroup  $\Delta$  which is not even finitely generated. A typical situation is when  $\Gamma$  contains a normal infinite amenable subgroup  $\Delta$ . Then all the  $L^2$ -Betti numbers of  $B\Gamma$  are trivial by a result of Cheeger and Gromov [11, Theorem 0.2 on page 191]. This result was the main motivation for our attempt to construct the extensions of dimension and of  $L^2$ -Betti numbers described above.

In Section 5 we will get the theorem of Cheeger and Gromov mentioned above as a corollary of the following result. If  $\Gamma$  is amenable and  $M$  is a  $\mathbb{C}\Gamma$ -module, then

$$\dim_{\mathcal{N}(\Gamma)} (\mathrm{Tor}_p^{\mathbb{C}\Gamma}(M, \mathcal{N}(\Gamma))) = 0 \quad \text{for } p \geq 1, \quad (0.9)$$

where we consider  $\mathcal{N}(\Gamma)$  as a  $\mathcal{N}(\Gamma)$ - $\mathbb{C}\Gamma$ -bimodule (Theorem 5.1). We get from 0.9 by a spectral sequence argument that the  $L^2$ -Betti numbers of a  $\Gamma$ -space  $X$  depend only on its singular homology with complex coefficients viewed as  $\mathbb{C}\Gamma$ -module, namely

$$b_p^{(2)}(X; \mathcal{N}(\Gamma)) = \dim_{\mathcal{N}(\Gamma)} (\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} H_p^{\mathrm{sing}}(X; \mathbb{C})), \quad (0.10)$$

provided that  $\Gamma$  is amenable (Theorem 5.11). The result of Cheeger and Gromov mentioned above follows from 0.10 since the singular homology of  $E\Gamma$  is trivial in all dimension except for dimension 0 where it is  $\mathbb{C}$ .

Equation 0.9 will also play a cruircial role in detecting non-trivial elements in the Grothendieck group  $G_0(\mathbb{C}\Gamma)$  of finitely generated  $\mathbb{C}\Gamma$ -modules for amenable groups  $\Gamma$  which will be investigated in Section 6. We will construct for amenable  $\Gamma$  a map

$$G_0(\mathbb{C}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \text{class}(\Gamma)_{cf}, \quad (0.11)$$

where  $\text{class}(\Gamma)_{cf}$  is the complex vector space of functions from the set  $\text{con}(\Gamma)_{cf}$  of finite conjugacy classes  $(\gamma)$  of elements in  $\Gamma$  to  $\mathbb{C}$  (Lemma 6.3 and Theorem 6.12). This map is related to the Hattori-Stallings rank and the universal center-valued trace and dimension of  $\mathcal{N}(\Gamma)$  (Theorem 6.12). In particular we will show that the class of  $\mathbb{C}\Gamma$  in  $G_0(\mathbb{C}\Gamma)$  generates an infinite cyclic direct summand in  $G_0(\mathbb{C}\Gamma)$  if  $\Gamma$  is amenable and is trivial if  $\Gamma$  contains a free group of rank 2 as subgroup (Remark 6.23).

We will investigate for  $d = 0, 1, \dots$  and  $d = \infty$  the class  $\mathcal{B}_d$  of groups  $\Gamma$  for which  $b_p(E\Gamma; \mathcal{N}(\Gamma)) = 0$  for  $p \leq d$  (Theorem 7.3) and discuss applications in Section 7 (Theorem 7.2).

We analyse  $L^2$ -Euler characteristics and the Burnside group in Section 8 generalizing the classical notions of Burnside ring, Burnside ring congruences and equivariant Euler characteristic for finite groups to infinite groups (Theorem 8.4, Lemma 8.11, Lemma 8.13, Remark 8.14 and Lemma 8.17). In particular the  $L^2$ -Euler characteristic extends the notion of virtual Euler characteristic of a group to a larger class of groups (Corollary 8.5).

In Section 9 we analyse the possible values of the  $L^2$ -Betti numbers (Theorem 9.2). If there is no bound on the orders of finite subgroups of  $\Gamma$ , then any non-negative real number can be realized as  $b_p^{(2)}(X; \mathcal{N}(\Gamma))$  for  $p \geq 3$  and a free  $\Gamma$ -CW-complex  $X$ . Otherwise we show for the least common multiple  $d$  of the orders of finite subgroups that  $d \cdot b_p^{(2)}(X; \mathcal{N}(\Gamma))$  is an integer or infinite for any  $\Gamma$ -space  $X$  if this holds for any finite free  $\Gamma$ -CW-complex  $Y$  (Theorem 9.2). The last conditions hold for instance for elementary amenable groups and free groups by Linnell [26].

The paper is organized as follows:

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1. Review of von Neumann dimension
2. The generalized dimension function
3. Induction for group von Neumann algebras
4.  $L^2$ -invariants for arbitrary  $\Gamma$ -spaces
5. Amenable groups
6. Dimension functions and  $G_0(\mathbb{C}\Gamma)$
7. Groups with vanishing  $L^2$ -Betti numbers
8.  $L^2$ -Euler characteristics and the Burnside group
9. Values of  $L^2$ -Betti numbers
- References

# 1. Review of von Neumann dimension

In this section we recall some basic facts about finitely generated Hilbert-modules and finitely generated projective modules over a finite von Neumann algebra. We fix for the sequel

**Notation 1.1** *Let  $\mathcal{A}$  be a finite von Neumann algebra and  $\text{tr} : \mathcal{A} \longrightarrow \mathbb{C}$  be a normal finite faithful trace. Denote by  $\Gamma$  an (arbitrary) discrete group. Let  $\mathcal{N}(\Gamma)$  be the group von Neumann algebra with the standard trace  $\text{tr}_{\mathcal{N}(\Gamma)}$ .*

*Module means always left-module and group actions on spaces are from the left unless explicitly stated differently. We will always work in the category of compactly generated spaces (see [43] and [47, I.4]).* ■

Next we recall our main example for  $\mathcal{A}$  and  $\text{tr}$ , namely *the group von Neumann algebra  $\mathcal{N}(\Gamma)$  with the standard trace*. The reader who is not familiar with the general concept of finite von Neumann algebras may always think of this example. Let  $l^2(\Gamma)$  be the Hilbert space of formal sums  $\sum_{\gamma \in \Gamma} \lambda_\gamma \cdot \gamma$  with complex coefficients  $\lambda_\gamma$  which are square-summable, i.e.  $\sum_{\gamma \in \Gamma} |\lambda_\gamma|^2 < \infty$ . Define *the group von Neumann algebra and the standard trace* by

$$\mathcal{N}(\Gamma) := \mathcal{B}(l^2(\Gamma), l^2(\Gamma))^\Gamma; \quad (1.2)$$

$$\text{tr}_{\mathcal{N}(\Gamma)}(a) := \langle a(e), e \rangle_{l^2(\Gamma)}; \quad (1.3)$$

where  $\mathcal{B}(l^2(\Gamma), l^2(\Gamma))^\Gamma$  is the space of bounded  $\Gamma$ -equivariant operators from  $l^2(\Gamma)$  to itself,  $a \in \mathcal{N}(\Gamma)$  and  $e \in \Gamma \subset l^2(\Gamma)$  is the unit element. The given trace on  $\mathcal{A}$  extends to a trace on square-matrices over  $\mathcal{A}$  in the usual way

$$\text{tr} : M(n, n, \mathcal{A}) \longrightarrow \mathbb{C} \quad A \mapsto \sum_{i=1}^n \text{tr}(A_{i,i}). \quad (1.4)$$

Taking adjoints induces the structure of a *ring with involution* on  $\mathcal{A}$ , i.e. we obtain a map  $*$  :  $\mathcal{A} \longrightarrow \mathcal{A}$   $a \mapsto a^*$ , which satisfies  $(a + b)^* = a^* + b^*$ ,  $(ab)^* = b^*a^*$  and  $(a^*)^* = a$  and  $1^* = 1$  for all  $a, b \in \mathcal{A}$ . This involution induces an involution on matrices

$$* : M(m, n, \mathcal{A}) \longrightarrow M(n, m, \mathcal{A}) \quad A = (A_{i,j}) \mapsto A^* = (A_{j,i}^*). \quad (1.5)$$

**Definition 1.6** *Let  $P$  be a finitely generated projective  $\mathcal{A}$ -module. Let  $A \in M(n, n, \mathcal{A})$  be a matrix such that  $A = A^*$ ,  $A^2 = A$  and the image of the  $\mathcal{A}$ -linear map  $A : \mathcal{A}^n \longrightarrow \mathcal{A}^n$  induced by right multiplication with  $A$  is  $\mathcal{A}$ -isomorphic to  $P$ . Define the von Neumann dimension of  $P$*

$$\dim(P) = \dim_{\mathcal{A}}(P) := \text{tr}_{\mathcal{A}}(A) \in [0, \infty). \quad \blacksquare$$

It is not hard to check that this definition is independent of the choice of  $A$  and depends only on the isomorphism class of  $P$ . Moreover the dimension is faithful, i.e.  $\dim(P) = 0$  implies  $P = 0$ , is additive under direct sums and satisfies  $\dim(\mathcal{A}^n) = n$ .

We recall that we have defined  $\overline{K}$ ,  $\mathbf{TM}$  and  $\mathbf{PM}$  for  $K \subset M$  in Definition 0.5. A sequence  $L \xrightarrow{f} M \xrightarrow{g} N$  of  $\mathcal{A}$ -modules is *weakly exact* resp. *exact at M* if  $\overline{\text{im}(f)} = \ker(g)$  resp.  $\text{im}(f) = \ker(g)$  holds. A morphism  $f : M \longrightarrow N$  of  $\mathcal{A}$ -modules is called a *weak isomorphism* if its kernel is trivial and the closure of its image is  $N$ .

Next we explain how these concepts above correspond to their analogues for finitely generated Hilbert  $\mathcal{A}$ -modules. Let  $l^2(\mathcal{A})$  be the Hilbert space completion of  $\mathcal{A}$  which is viewed as a pre-Hilbert space by the inner product  $\langle a, b \rangle = \text{tr}(ab^*)$ . A *finitely generated Hilbert  $\mathcal{A}$ -module*  $V$  is a Hilbert space  $V$  together with a left operation of  $\mathcal{A}$  by  $\mathbb{C}$ -linear maps such that there exists a unitary  $\mathcal{A}$ -embedding of  $V$  in  $\oplus_{i=1}^n l^2(\mathcal{A})$  for some  $n$ . A morphism of finitely generated Hilbert  $\mathcal{A}$ -modules is a bounded  $\mathcal{A}$ -equivariant operator. Denote by  $\{\text{fin. gen. Hilb. } \mathcal{A}\text{-mod.}\}$  the category of finitely generated Hilbert  $\mathcal{A}$ -modules. A sequence  $U \xrightarrow{f} V \xrightarrow{g} W$  of finitely generated Hilbert  $\mathcal{A}$ -modules is *exact* resp. *weakly exact at V* if  $\text{im}(f) = \ker(g)$  resp.  $\overline{\text{im}(f)} = \ker(g)$  holds. A morphism  $f : V \longrightarrow W$  is a *weak isomorphism* if its kernel is trivial and its image is dense. For a survey on finite von Neumann algebras and Hilbert  $\mathcal{A}$ -modules we refer for instance to [29, section 1], [36, section 1].

The right regular representation  $\mathcal{A} \longrightarrow \mathcal{B}(l^2(\mathcal{A}), l^2(\mathcal{A}))^{\mathcal{A}}$  from  $\mathcal{A}$  into the space of bounded  $\mathcal{A}$ -equivariant operators from  $l^2(\mathcal{A})$  to itself sends  $a \in \mathcal{A}$  to the extension of the map  $\mathcal{A} \longrightarrow \mathcal{A} \quad b \mapsto ba^*$ . It is known to be bijective [15, Theorem 1 in I.5.2 on page 80, Theorem 2 in I.6.2 on page 99]. Hence we obtain a bijection

$$\nu : M(m, n, \mathcal{A}) \longrightarrow \mathcal{B}(l^2(\mathcal{A})^m, l^2(\mathcal{A})^n)^{\mathcal{A}}, \quad (1.7)$$

which is compatible with the  $\mathbb{C}$ -vector space structures, the involutions and composition.

The details of the following theorem and its proof can be found in [34, section 2]. It is essentially a consequence of 1.7 and the construction of the idempotent completion of a category. It allows us to forget the Hilbert-module-structures and simply work with the von Neumann algebra as a plain ring. An equivalent approach is given by Farber [20],[21] and is identified with the one here in [34, Theorem 0.9]. An *inner product* on a finitely generated projective  $\mathcal{A}$ -module  $P$  is a map  $\mu : P \times P \longrightarrow \mathcal{A}$  which is linear in the first variable, symmetric in the sense  $\mu(x, y) = \mu(y, x)^*$  and positive in the sense  $\mu(x, x) > 0 \iff x \neq 0$  such that the induced map  $P \longrightarrow P^*$  sending  $y \in P$  to  $\mu(-, y)$  is bijective.

**Theorem 1.8**    1. *There is a functor*

$$\nu : \{\text{fin. gen. proj. } \mathcal{A}\text{-mod. with inner prod.}\} \longrightarrow \{\text{fin. gen. Hilb. } \mathcal{A}\text{-mod.}\}$$

*which is an equivalence of  $\mathbb{C}$ -categories with involutions;*

2. *Any finitely generated projective  $\mathcal{A}$ -module has an inner product. Two finitely generated projective  $\mathcal{A}$ -modules with inner product are unitarily  $\mathcal{A}$ -isomorphic if and only if the underlying  $\mathcal{A}$ -modules are  $\mathcal{A}$ -isomorphic;*

3. *Let  $\nu^{-1}$  be an inverse of  $\nu$  which is well-defined up to unitary natural equivalence. The composition of  $\nu^{-1}$  with the forgetful functor induces an equivalence of  $\mathbb{C}$ -categories*

$$\{\text{fin. gen. Hilb. } \mathcal{A}\text{-mod.}\} \longrightarrow \{\text{fin. gen. proj. } \mathcal{A}\text{-mod.}\};$$

4.  $\nu$  and  $\nu^{-1}$  preserve weak exactness and exactness. ■

Of course Definition 1.6 of  $\dim(P)$  for a finitely generated projective  $\mathcal{A}$ -module agrees with the usual von Neumann dimension of the associated Hilbert  $\mathcal{A}$ -module  $\nu(P)$  after any choice of inner product on  $P$ .

## 2. The generalized dimension function

In this section we give the proof of Theorem 0.6 and investigate the behaviour of the dimension under colimits. We recall that we have introduced  $\dim(M)$  for an arbitrary  $\mathcal{A}$ -module  $M$  in Definition 0.4 and  $\overline{K}$ ,  $\mathbf{T}M$  and  $\mathbf{P}M$  for  $K \subset M$  in Definition 0.5. We begin with the proof of Theorem 0.6.

**Proof :** 1.) is proven in [34, Corollary 2.4] for finite von Neumann algebras. However, Pardo pointed out to us that any von Neumann algebra is semi-hereditary. This follows from the facts that any von Neumann algebra is a Baer  $*$ -ring and hence in particular a Rickart  $C^*$ -algebra [6, Definition 1, Definition 2 and Proposition 9 in Chapter 1.4] and that a  $C^*$ -algebra is semi-hereditary if and only if it is Rickart [1, Corollary 3.7 on page 270].

2.) and 4.a) in the special case that  $M = P$  for a finitely generated projective  $\mathcal{A}$ -module  $P$ .

Let  $\mathcal{P} = \{P_i \mid i \in I\}$  be the directed system of finitely generated projective  $\mathcal{A}$ -modules of  $K$ . Notice that  $\mathcal{P}$  is indeed directed by inclusion since the submodule of  $P$  generated by two finitely generated projective submodules is again finitely generated and hence by 1.) finitely generated projective. Let  $j_i : P_i \longrightarrow P$  be the inclusion. Equip  $P$  and each  $P_i$  with a fixed inner product and let  $\text{pr}_i : \nu(P) \longrightarrow \nu(P)$  be the orthogonal projection satisfying  $\text{im}(\text{pr}_i) = \overline{\text{im}(\nu(j_i))}$  and  $\text{pr} : \nu(P) \longrightarrow \nu(P)$  be the orthogonal projection satisfying  $\text{im}(\text{pr}) = \bigcup_{i \in I} \text{im}(\text{pr}_i)$ . Next we show

$$\text{im}(\nu^{-1}(\text{pr})) = \overline{K}. \quad (2.1)$$

Let  $f : P \longrightarrow \mathcal{A}$  be a  $\mathcal{A}$ -map with  $K \subset \ker(f)$ . Then  $f \circ j_i = 0$  and therefore  $\nu(f) \circ \nu(j_i) = 0$  for all  $i \in I$ . We get  $\text{im}(\text{pr}_i) \subset \ker(\nu(f))$  for all  $i \in I$ . Because the kernel of  $\nu(f)$  is closed we conclude  $\text{im}(\text{pr}) \subset \ker(\nu(f))$ . This shows  $\text{im}(\nu^{-1}(\text{pr})) \subset \ker(f)$  and hence  $\text{im}(\nu^{-1}(\text{pr})) \subset \overline{K}$ . As  $K \subset \ker(\text{id} - \nu^{-1}(\text{pr})) = \text{im}(\nu^{-1}(\text{pr}))$ , we conclude  $\overline{K} \subset \text{im}(\nu^{-1}(\text{pr}))$ . This finishes the proof of 2.1 and of 2.) in the special case  $M = P$ .

Next we prove

$$\dim'(K) = \dim(\overline{K}). \quad (2.2)$$

The inclusion  $j_i$  induces a weak isomorphism  $\nu(P_i) \longrightarrow \text{im}(\text{pr}_i)$  of finitely generated Hilbert  $\mathcal{A}$ -modules. If we apply the Polar Decomposition Theorem to it we obtain a unitary  $\mathcal{A}$ -isomorphism from  $\nu(P_i)$  to  $\text{im}(\text{pr}_i)$ . This implies  $\dim(P_i) = \text{tr}(\text{pr}_i)$ . Therefore it remains to prove

$$\text{tr}(\text{pr}) := \sup\{\text{tr}(\text{pr}_i) \mid i \in I\}. \quad (2.3)$$



As  $\text{tr}$  is normal, it suffices to show for  $x \in \nu(P)$  that the net  $\{\text{pr}_i(x) \mid i \in I\}$  converges to  $\text{pr}(x)$ . Let  $\epsilon > 0$  be given. Choose  $i(\epsilon) \in I$  and  $x_{i(\epsilon)} \in \text{im}(\text{pr}_{i(\epsilon)})$  with  $\|\text{pr}(x) - x_{i(\epsilon)}\| \leq \epsilon/2$ . We conclude for all  $i \geq i(\epsilon)$

$$\begin{aligned}
\|\text{pr}(x) - \text{pr}_i(x)\| &\leq \|\text{pr}(x) - \text{pr}_{i(\epsilon)}(x)\| \\
&\leq \|\text{pr}(x) - \text{pr}_{i(\epsilon)}(x_{i(\epsilon)})\| + \|\text{pr}_{i(\epsilon)}(x_{i(\epsilon)}) - \text{pr}_{i(\epsilon)}(x)\| \\
&\leq \|\text{pr}(x) - x_{i(\epsilon)}\| + \|\text{pr}_{i(\epsilon)}(x_{i(\epsilon)} - \text{pr}(x))\| \\
&\leq \|\text{pr}(x) - x_{i(\epsilon)}\| + \|\text{pr}_{i(\epsilon)}\| \cdot \|x_{i(\epsilon)} - \text{pr}(x)\| \\
&\leq 2 \cdot \|\text{pr}(x) - x_{i(\epsilon)}\| \\
&\leq \epsilon.
\end{aligned}$$

Now 2.3 and hence 2.2 follow. In particular we get from 2.2 for any finitely generated projective submodule  $Q_0$  of a finitely generated projective  $\mathcal{A}$ -module  $Q$

$$\dim(Q_0) \leq \dim(Q), \quad (2.4)$$

since by definition  $\dim(Q_0) \leq \dim'(Q_0)$  and  $\dim(\overline{Q_0}) \leq \dim(Q)$  follows from additivity of  $\dim$  under direct sums and that we have already proven that  $\overline{Q_0}$  is a direct summand in  $Q$ . This implies for a finitely generated projective  $\mathcal{A}$ -module  $Q$

$$\dim(Q) = \dim'(Q). \quad (2.5)$$

Now 2.2 and 2.5 imply 4.a.) in the special case that  $M = P$  for a finitely generated projective  $\mathcal{A}$ -module  $P$ .

4.d.) has been already proven in 2.5.

4.b.) If  $P \subset M$  is a finitely generated projective submodule, then there is an index  $i \in I$  with  $P \subset M_i$  by cofinality.

4.c.) Let  $P \subset M_2$  be a finitely generated projective submodule. We obtain an exact sequence  $0 \longrightarrow M_0 \longrightarrow p^{-1}(P) \longrightarrow P \longrightarrow 0$ . Since  $p^{-1}(P) \cong M_0 \oplus P$ , we conclude

$$\dim'(M_0) + \dim(P) \leq \dim'(p^{-1}(P)) \leq \dim'(M_1).$$

Since this holds for all finitely generated projective submodules  $P \subset M_2$ , we get

$$\dim'(M_0) + \dim'(M_2) \leq \dim'(M_1). \quad (2.6)$$

Let  $Q \subset M_1$  be finitely generated projective. We obtain exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & i(M_0) \cap Q & \longrightarrow & Q & \longrightarrow & p(Q) \longrightarrow 0; \\
0 & \longrightarrow & \overline{i(M_0) \cap Q} & \longrightarrow & Q & \longrightarrow & Q/\overline{i(M_0) \cap Q} \longrightarrow 0.
\end{array}$$

By the special case of 2.) which we have already proven above  $\overline{i(M_0) \cap Q}$  is a direct summand in  $Q$ . We conclude

$$\dim(Q) = \dim(\overline{i(M_0) \cap Q}) + \dim(Q/\overline{i(M_0) \cap Q}).$$

From the special case 4.a.) we have already proven above, 4.d.) and the fact that there is an epimorphism from  $p(Q)$  onto the finitely generated projective  $\mathcal{A}$ -module  $Q/\overline{i(M_0) \cap Q}$ , we conclude

$$\begin{aligned} \dim(\overline{i(M_0) \cap Q}) &= \dim'(i(M_0) \cap Q); \\ \dim(Q/\overline{i(M_0) \cap Q}) &\leq \dim'(p(Q)). \end{aligned}$$

Since obviously  $\dim'(M) \leq \dim'(N)$  holds for  $\mathcal{A}$ -modules  $M$  and  $N$  with  $M \subset N$ , we get

$$\begin{aligned} \dim(Q) &= \dim(\overline{i(M_0) \cap Q}) + \dim(Q/\overline{i(M_0) \cap Q}) \\ &\leq \dim'(i(M_0) \cap Q) + \dim'(p(Q)) \\ &\leq \dim'(M_0) + \dim'(M_2). \end{aligned}$$

Since this holds for all finitely generated projective submodules  $Q \subset M_1$ , we get

$$\dim'(M_1) \leq \dim'(M_0) + \dim'(M_2). \quad (2.7)$$

Now 4.c.) follows from 2.6 and 2.7.

2.) and 4.a.) Choose a finitely generated free  $\mathcal{A}$ -module  $F$  together with an epimorphism  $q : F \rightarrow M$ . One easily checks that  $q^{-1}(\overline{K})$  is  $\overline{q^{-1}(K)}$  and that  $F/q^{-1}(\overline{K})$  and  $M/\overline{K}$  are isomorphic. From the special case of 2.) and 4.) a.) which we have already proven above we conclude that  $F/\overline{q^{-1}(K)}$  and hence  $M/\overline{K}$  are finitely generated projective and

$$\dim'(q^{-1}(K)) = \dim'(\overline{q^{-1}(K)}) = \dim'(q^{-1}(\overline{K})).$$

If  $L$  is the kernel of  $q$ , we conclude from Additivity

$$\begin{aligned} \dim'(q^{-1}(\overline{K})) &= \dim'(L) + \dim'(\overline{K}); \\ \dim'(q^{-1}(K)) &= \dim'(L) + \dim'(K). \end{aligned}$$

Now 2.) and 4.a) follow in general.

3.) follows from 2.), as  $\overline{\{0\}} = \mathbf{T}M$  and  $M/\mathbf{T}M = \mathbf{P}M$  by definition.

4.e.) From 2.), 4.c.) and 4.d.) we get:  $\dim'(M) = \dim'(\mathbf{T}M) + \dim(\mathbf{P}M)$ . If we apply 4.a) to  $\{0\} \subset M$  we get  $\dim'(\mathbf{T}M) = 0$  because of  $\overline{\{0\}} = \mathbf{T}M$ .

4.f.) Let  $\dim''$  be another function satisfying Continuity, Cofinality, Additivity and the Extension Property. We want to show for a  $\mathcal{A}$ -module  $M$

$$\dim''(M) = \dim'(M).$$

Since 4.e.) is a consequence of Continuity, Additivity and the Extension Property alone, this is obvious provided  $M$  is finitely generated. Since the system of finitely generated submodules of a module is cofinal, the claim follows from Cofinality. This finishes the proof of Theorem 0.6. ■

**Notation 2.8** In view of Theorem 0.6 we will not distinguish between  $\dim'$  and  $\dim$  in the sequel. ■

Next we investigate the behaviour of dimension under colimits indexed by a directed set. We mention that colimit is sometimes called in the literature also inductive limit or direct limit. The harder case of inverse limits which is not needed in this paper will be treated at a different place (see also [11, Appendix]).

**Theorem 2.9** *Let  $I$  be a category such that between two objects there is at most one morphism and for two objects  $i_1$  and  $i_2$  there is an object  $i_0$  with  $i_1 \leq i_0$  and  $i_2 \leq i_0$  where we write  $i \leq k$  for two objects  $i$  and  $k$  if and only if there is a morphism from  $i$  to  $k$ . Let  $M_i$  be a covariant functor from  $I$  to the category of  $\mathcal{A}$ -modules. For  $i \leq j$  let  $\phi_{i,j} : M_i \longrightarrow M_j$  be the associated morphism of  $\mathcal{A}$ -modules. For  $i \in I$  let  $\psi_i : M_i \longrightarrow \operatorname{colim}_I M_i$  be the canonical morphism of  $\mathcal{A}$ -modules. Then:*

1. *We get for the dimension of the  $\mathcal{A}$ -module given by the colimit  $\operatorname{colim}_I M_i$*

$$\dim(\operatorname{colim}_I M_i) = \sup \{ \dim(\operatorname{im}(\psi_i)) \mid i \in I \};$$

2. *Suppose for each  $i \in I$  that there is  $i_0 \in I$  with  $i \leq i_0$  such that  $\dim(\operatorname{im}(\phi_{i,i_0})) < \infty$  holds. Then:*

$$\dim(\operatorname{colim}_I M_i) = \sup \{ \inf \{ \dim(\operatorname{im}(\phi_{i,j} : M_i \longrightarrow M_j)) \mid j \in I, i \leq j \} \mid i \in I \}.$$

**Proof :** 1.) Recall that  $\operatorname{colim}_I M_i$  can be constructed as  $\coprod_{i \in I} M_i / \sim$  for the equivalence relation for which  $x \in M_i \sim y \in M_j$  holds precisely if there is  $k \in I$  with  $i \leq k$  and  $j \leq k$  with the property  $\phi_{i,k}(x) = \phi_{j,k}(y)$ . With this description one easily checks

$$\operatorname{colim}_I M_i = \cup_{i \in I} \operatorname{im}(\psi_i : M_i \longrightarrow \operatorname{colim}_I M_i)$$

Now apply Cofinality of  $\dim$  (see Theorem 0.6.4).

2.) It remains to show for  $i \in I$

$$\dim(\operatorname{im}(\psi_i)) = \inf \{ \dim(\operatorname{im}(\phi_{i,j} : M_i \longrightarrow M_j)) \mid j \in I, i \leq j \}. \quad (2.10)$$

By assumption there is  $i_0 \in I$  with  $i \leq i_0$  such that  $\dim(\operatorname{im}(\phi_{i,i_0}))$  is finite. Let  $K_{i_0,j}$  be the kernel of the map  $\operatorname{im}(\phi_{i,i_0}) \longrightarrow \operatorname{im}(\phi_{i,j})$  induced by  $\phi_{i_0,j}$  for  $i_0 \leq j$  and  $K_{i_0}$  be the kernel of the map  $\operatorname{im}(\phi_{i,i_0}) \longrightarrow \operatorname{im}(\psi_i)$  induced by  $\psi_{i_0}$ . Then  $K_{i_0} = \cup_{j \in I, i_0 \leq j} K_{i_0,j}$  and hence by Cofinality (see Theorem 0.6.4)

$$\dim(K_{i_0}) = \sup \{ \dim(K_{i_0,j}) \mid j \in I, i_0 \leq j \}.$$

Since  $\dim(\operatorname{im}(\phi_{i,i_0}))$  is finite, we get from Additivity (see Theorem 0.6.4)

$$\begin{aligned} \dim(\operatorname{im}(\psi_i)) &= \dim(\operatorname{im}(\psi_{i_0}|_{\operatorname{im}(\phi_{i,i_0})} : \operatorname{im}(\phi_{i,i_0}) \longrightarrow \operatorname{colim}_I M_i)) \\ &= \dim(\operatorname{im}(\phi_{i,i_0})) - \dim(K_{i_0}) \\ &= \dim(\operatorname{im}(\phi_{i,i_0})) - \sup \{ \dim(K_{i_0,j}) \mid j \in I, i_0 \leq j \} \\ &= \inf \{ \dim(\operatorname{im}(\phi_{i,i_0})) - \dim(K_{i_0,j}) \mid j \in I, i_0 \leq j \} \\ &= \inf \{ \dim(\operatorname{im}(\phi_{i_0,j}|_{\operatorname{im}(\phi_{i,i_0})} : \operatorname{im}(\phi_{i,i_0}) \longrightarrow \operatorname{im}(\phi_{i,j}))) \mid j \in I, i_0 \leq j \} \\ &= \inf \{ \dim(\operatorname{im}(\phi_{i,j})) \mid j \in I, i_0 \leq j \} \end{aligned} \quad (2.11)$$

Given  $j_0 \in J$  with  $i \leq j_0$ , there is  $j \in I$  with  $i_0 \leq j$  and  $j_0 \leq j$  and hence with

$$\dim(\text{im}(\phi_{i,j_0})) \geq \dim(\text{im}(\phi_{i,j})).$$

This implies

$$\inf\{\dim(\text{im}(\phi_{i,j})) \mid j \in J, i \leq j\} = \inf\{\dim(\text{im}(\phi_{i,j})) \mid j \in J, i_0 \leq j\}. \quad (2.12)$$

Now 2.10 follows from 2.11 and 2.12. This finishes the proof of Theorem 2.9.  $\blacksquare$

**Example 2.13** The condition in Theorem 2.9.2 that for each  $i \in I$  there is  $i_0 \in I$  with  $i \leq i_0$  with  $\dim(\text{im}(\phi_{i,i_0})) < \infty$  is necessary as the following example shows. Take  $I = \mathbb{N}$ . Define  $M_j = \bigoplus_{n=j}^{\infty} \mathcal{A}$  and  $\phi_{j,k} : \bigoplus_{m=j}^{\infty} \mathcal{A} \longrightarrow \bigoplus_{m=k}^{\infty} \mathcal{A}$  to be the projection. Then  $\dim(\text{im}(\phi_{j,k})) = \infty$  for all  $j \leq k$ , but  $\text{colim}_I M_i$  is trivial and hence has dimension zero.  $\blacksquare$

**Remark 2.14** From an axiomatic point of view we have only needed the following basic properties of  $\mathcal{A}$ . Namely, let  $R$  be an associative ring with unit which has the following properties

1. There is a dimension function  $\dim$  which assigns to any finitely generated projective  $R$ -module  $P$  an element

$$\dim(P) \in [0, \infty)$$

such that  $\dim(P \oplus Q) = \dim(P) + \dim(Q)$  holds and  $\dim(P)$  depends only on the isomorphism class of  $P$ ;

2. If  $K \subset P$  is a submodule of the finitely generated projective  $\mathcal{A}$ -module  $P$ , then  $\overline{K}$  is a direct summand in  $P$ . Moreover

$$\dim(\overline{K}) = \sup\{\dim(P) \mid P \subset K \text{ finitely generated projective } R\text{-submodule}\}.$$

Then with Definition 0.4 Theorem 0.6 carries over to  $R$ . One has essentially to copy the part of the proof which begins with 2.4.

An easy example where these axioms are satisfied is the case where  $R$  is a principal ideal domain and  $\dim$  is the usual rank of a finitely generated free  $R$ -module. Then the extended dimension for a  $R$ -module  $M$  is just the dimension of the rational vector space  $F \otimes_R M$  for  $F$  the quotient field of  $R$ . Notice that the case of a von Neumann algebra  $R = \mathcal{A}$  is harder since  $\mathcal{A}$  is not noetherian in general.

In Definition 0.5 we have defined  $\mathbf{TM}$  and  $\mathbf{PM}$  only for finitely generated  $\mathcal{A}$ -modules  $M$  although the definition makes sense in general. The reason is that the following definition for arbitrary  $\mathcal{A}$ -modules seems to be more appropriate

$$\mathbf{TM} := \bigcup \{N \subset M \mid \dim(N) = 0\}; \quad (2.15)$$

$$\mathbf{PM} := M/\mathbf{TM}. \quad (2.16)$$

One easily checks using Theorem 0.6.4 that  $\mathbf{TM}$  is the largest submodule of  $M$  with trivial dimension and that these definitions 2.15 and 2.16 agree with Definition 0.5 if  $M$  is finitely generated. One can show by example that they do not agree if one applies them to arbitrary  $\mathcal{A}$ -modules. In the case of a principal ideal domain  $R$  the torsion submodule of a  $R$ -module  $M$  is just  $\mathbf{TM}$  in the sense of definition 2.15.  $\blacksquare$

### 3. Induction for group von Neumann algebras

Next we investigate how the dimension behaves under induction. Let  $i : \Delta \longrightarrow \Gamma$  be an inclusion of groups. We claim that associated to  $i$  there is a ring homomorphism of the group von Neumann algebras, also denoted by

$$i : \mathcal{N}(\Delta) \longrightarrow \mathcal{N}(\Gamma). \quad (3.1)$$

Recall that  $\mathcal{N}(\Delta)$  is the same as the ring  $\mathcal{B}(l^2(\Delta), l^2(\Delta))^\Delta$  of bounded  $\Delta$ -equivariant operators  $f : l^2(\Delta) \longrightarrow l^2(\Delta)$ . Notice that  $\mathbb{C}\Gamma \otimes_{\mathbb{C}\Delta} l^2(\Delta)$  can be viewed as a dense subspace of  $l^2(\Gamma)$  and that  $f$  defines a  $\mathbb{C}\Gamma$ -homomorphism  $\text{id} \otimes_{\mathbb{C}\Delta} f : \mathbb{C}\Gamma \otimes_{\mathbb{C}\Delta} l^2(\Delta) \longrightarrow \mathbb{C}\Gamma \otimes_{\mathbb{C}\Delta} l^2(\Delta)$  which is bounded with respect to the pre-Hilbert structure induced on  $\mathbb{C}\Gamma \otimes_{\mathbb{C}\Delta} l^2(\Delta)$  from  $l^2(\Gamma)$ . Hence  $\text{id} \otimes_{\mathbb{C}\Delta} f$  extends to a  $\Gamma$ -equivariant bounded operator  $i(f) : l^2(\Gamma) \longrightarrow l^2(\Gamma)$ .

Given a  $\mathcal{N}(\Delta)$ -module  $M$ , define *the induction with  $i$*  to be the  $\mathcal{N}(\Gamma)$ -module

$$i_*(M) := \mathcal{N}(\Gamma) \otimes_{\mathcal{N}(\Delta)} M. \quad (3.2)$$

Obviously  $i_*$  is a covariant functor from the category of  $\mathcal{N}(\Delta)$ -modules to the category of  $\mathcal{N}(\Gamma)$ -modules, preserves direct sums and the properties finitely generated and projective and sends  $\mathcal{N}(\Delta)$  to  $\mathcal{N}(\Gamma)$ .

**Theorem 3.3** *Let  $i : \Delta \longrightarrow \Gamma$  be an injective group homomorphisms. Then:*

1. *Induction with  $i$  is a faithfully flat functor from the category of  $\mathcal{N}(\Delta)$ -modules to the category of  $\mathcal{N}(\Gamma)$ -modules, i.e. a sequence of  $\mathcal{N}(\Delta)$ -modules  $M_0 \longrightarrow M_1 \longrightarrow M_2$  is exact at  $M_1$  if and only if the induced sequence of  $\mathcal{N}(\Gamma)$ -modules  $i_*M_0 \longrightarrow i_*M_1 \longrightarrow i_*M_2$  is exact at  $i_*M_1$ ;*
2. *For any  $\mathcal{N}(\Delta)$ -module  $M$  we have:*

$$\dim_{\mathcal{N}(\Delta)}(M) = \dim_{\mathcal{N}(\Gamma)}(i_*M).$$

**Proof** : The proof consists of the following steps.

Step 1:  $\dim_{\mathcal{N}(\Delta)}(M) = \dim_{\mathcal{N}(\Gamma)}(i_*(M))$ , provided  $M$  is a finitely generated projective  $\mathcal{N}(\Delta)$ -module.

Let  $A \in M(n, n, \mathcal{N}(\Delta))$  be a matrix such that  $A = A^*$ ,  $A^2 = A$  and the image of the  $\mathcal{N}(\Delta)$ -linear map  $A : \mathcal{N}(\Delta)^n \longrightarrow \mathcal{N}(\Delta)^n$  induced by right multiplication with  $A$  is  $\mathcal{N}(\Delta)$ -isomorphic to  $M$ . Let  $i(A)$  be the matrix in  $M(n, n, \mathcal{N}(\Gamma))$  obtained from  $A$  by applying  $i$  to each entry. Then  $i(A) = i(A)^*$ ,  $i(A)^2 = i(A)$  and the image of the  $\mathcal{N}(\Gamma)$ -linear map  $i(A) : \mathcal{N}(\Gamma)^n \longrightarrow \mathcal{N}(\Gamma)^n$  induced by right multiplication with  $i(A)$  is  $\mathcal{N}(\Gamma)$ -isomorphic to  $i_*M$ . Hence we get from Definition 1.6

$$\begin{aligned} \dim_{\mathcal{N}(\Delta)}(M) &= \text{tr}_{\mathcal{N}(\Delta)}(A); \\ \dim_{\mathcal{N}(\Gamma)}(i_*M) &= \text{tr}_{\mathcal{N}(\Gamma)}(i(A)). \end{aligned}$$

Therefore it suffices to show  $\text{tr}_{\mathcal{N}(\Gamma)}(i(a)) = \text{tr}_{\mathcal{N}(\Delta)}(a)$  for  $a \in \mathcal{N}(\Delta)$ . This is an easy consequence of the Definition 1.3 of the standard trace.

Step 2: If  $M$  is finitely presented  $\mathcal{N}(\Delta)$ -module, then

$$\begin{aligned}\dim_{\mathcal{N}(\Delta)}(M) &= \dim_{\mathcal{N}(\Gamma)}(i_*(M)); \\ \text{Tor}_1^{\mathcal{N}(\Delta)}(\mathcal{N}(\Gamma), M) &= 0.\end{aligned}$$

Since  $M$  is finitely presented, it splits as  $M = \mathbf{T}M \oplus \mathbf{P}M$  where  $\mathbf{P}M$  is finitely generated projective and there is an exact sequence  $0 \rightarrow \mathcal{N}(\Delta)^n \xrightarrow{f} \mathcal{N}(\Delta)^n \rightarrow \mathbf{T}M \rightarrow 0$  with  $f^* = f$  [34, Theorem 1.2, Lemma 3.4]. If we apply the right exact functor induction with  $i$  to it, we get an exact sequence  $\mathcal{N}(\Gamma)^n \xrightarrow{i_*f} \mathcal{N}(\Gamma)^n \rightarrow i_*\mathbf{T}M \rightarrow 0$  with  $(i_*f)^* = i_*f$ . Because of Step 1, Additivity (see Theorem 0.6.4) and the definition of Tor it suffices to show that  $i_*f$  is injective. Let  $\nu$  be the functor introduced in Theorem 1.8 or [34, section 2]. Then  $i(\nu(f))$  is  $\nu(i_*f)$ . Because  $\nu$  respects weak exactness (see Theorem 1.8 or [34, Lemma 2.3])  $\nu(f)$  has dense image since  $\mathcal{N}(\Delta)^n \xrightarrow{i_*f} \mathcal{N}(\Delta)^n \rightarrow 0$  is weakly exact. Then one easily checks that  $\nu(i_*f) = i_*(\nu(f))$  has dense image since  $\mathbb{C}\Gamma \otimes_{\mathbb{C}\Delta} l^2(\Delta)$  is a dense subspace of  $l^2(\Gamma)$ . Since the kernel of a bounded operator of Hilbert spaces is the orthogonal complement of the image of its adjoint and  $\nu(i_*f)$  is selfadjoint,  $\nu(i_*f)$  is injective. Since  $\nu^{-1}$  respects exactness (see Theorem 1.8 or [34, Lemma 2.3])  $i_*f$  is injective.

Step 3:  $\text{Tor}_1^{\mathcal{N}(\Delta)}(\mathcal{N}(\Gamma), M) = 0$  provided,  $M$  is a finitely generated  $\mathcal{N}(\Delta)$ -module.

Choose an exact sequence  $0 \rightarrow K \xrightarrow{g} P \rightarrow M \rightarrow 0$  such that  $P$  is a finitely generated projective  $\mathcal{N}(\Delta)$ -module. The associated long exact sequence of Tor-groups shows that  $\text{Tor}_1^{\mathcal{N}(\Delta)}(\mathcal{N}(\Gamma), M)$  is trivial if and only if  $i_*g : i_*K \rightarrow i_*P$  is injective. For each element  $x$  in  $i_*K$  there is a finitely generated submodule  $K' \subset K$  such that  $x$  lies in the image of the map  $i_*K' \rightarrow i_*K$  induced by the inclusion. Hence it suffices to show for any finitely generated submodule  $K' \subset P$  that the inclusion induces an injection  $i_*K' \rightarrow i_*P$ . This follows since Step 2 applied to the finitely presented module  $P/K$  shows  $\text{Tor}_1^{\mathcal{N}(\Delta)}(\mathcal{N}(\Gamma), P/K) = 0$ .

Step 4:  $i_*$  is an exact functor.

By standard homological algebra we have to show that  $\text{Tor}_{\mathcal{N}(\Delta)}^1(\mathcal{N}(\Gamma), M) = 0$  is trivial for all  $\mathcal{N}(\Delta)$ -modules  $M$ . Notice that  $M$  is the colimit of the directed system of its finitely generated submodules (directed by inclusion) and that the functor Tor commutes in both variables with colimits over directed systems [10, Proposition VI.1.3. on page 107]. Now the claim follows from Step 3.

Step 5: Let  $\{M_i \mid i \in I\}$  be the directed system of finitely generated submodules of the  $\mathcal{N}(\Delta)$ -module  $M$ . Then

$$\begin{aligned}\dim_{\mathcal{N}(\Delta)}(M) &= \sup\{\dim_{\mathcal{N}(\Delta)}(M_i) \mid i \in I\}; \\ \dim_{\mathcal{N}(\Gamma)}(i_*M) &= \sup\{\dim_{\mathcal{N}(\Gamma)}(i_*M_i) \mid i \in I\}.\end{aligned}$$

Because of Step 4 we can view  $i_*M_i$  as a submodule of  $i_*M$ . Now apply Cofinality (see Theorem 0.6.4).

Step 6: The second assertion of Theorem 3.3 is true.

Because of Step 5 it suffices to prove the claim in the case that  $M$  is finitely generated because any module is the colimit of the directed system of its finitely generated submodules. Choose an exact sequence  $0 \longrightarrow K \xrightarrow{g} P \longrightarrow M \longrightarrow 0$  such that  $P$  is a finitely generated projective  $\mathcal{N}(\Delta)$ -module. Because of Step 4 and Additivity (see Theorem 0.6) we get

$$\begin{aligned}\dim_{\mathcal{N}(\Delta)}(M) &= \dim_{\mathcal{N}(\Delta)}(P) - \dim_{\mathcal{N}(\Delta)}(K); \\ \dim_{\mathcal{N}(\Gamma)}(i_*M) &= \dim_{\mathcal{N}(\Gamma)}(i_*P) - \dim_{\mathcal{N}(\Gamma)}(i_*K).\end{aligned}$$

Because of Step 1 it remains to prove

$$\dim_{\mathcal{N}(\Delta)}(K) = \dim_{\mathcal{N}(\Gamma)}(i_*K).$$

Because of Step 5 it suffices to treat the case where  $K \subset P$  is finitely generated. Since  $\mathcal{N}(\Delta)$  is semi-hereditary (see Theorem 0.6.1)  $K$  is finitely generated projective and the claim follows from Step 1.

Step 7: The first assertion of Theorem 3.3 is true.

Because we know already from Step 4 that  $i_*$  is exact, it remains to prove for a  $\mathcal{N}(\Delta)$ -module  $M$

$$i_*M = 0 \iff M = 0.$$

Suppose  $i_*M = 0$ . In order to show  $M = 0$  we have to prove for any  $\mathcal{N}(\Delta)$ -map  $f : \mathcal{N}(\Delta) \longrightarrow M$  that it is trivial. Let  $K$  be the kernel of  $f$ . Because  $i_*$  is exact by Step 4 and  $i_*M = 0$  by assumption, the inclusion induces an isomorphism  $i_*K \longrightarrow i_*\mathcal{N}(\Delta)$ . Since  $i_*K$  is a finitely generated  $\mathcal{N}(\Gamma)$ -module and  $i_*$  is exact by Step 4, there is a finitely generated submodule  $K' \subset K$  such that the inclusion induces an isomorphism  $i_*K' \longrightarrow i_*\mathcal{N}(\Delta)$ . Let  $\mathcal{N}(\Delta)^m \longrightarrow K'$  be an epimorphism. Let  $g : \mathcal{N}(\Delta)^m \longrightarrow \mathcal{N}(\Delta)$  be the obvious composition. Because  $i_*$  is exact by Step 4 the induced map  $i_*g : i_*\mathcal{N}(\Delta)^m \longrightarrow i_*\mathcal{N}(\Delta)$  is surjective. Hence it remains to prove that  $g$  itself is surjective because then  $K' = \mathcal{N}(\Delta)$  and the map  $f : \mathcal{N}(\Delta) \longrightarrow M$  is trivial. Since the functors  $\nu^{-1}$  and  $\nu$  of Theorem 1.8 are exact we have to show for a  $\Delta$ -equivariant bounded operator  $h : l^2(\Delta)^m \longrightarrow l^2(\Delta)$  that  $h$  is surjective if  $i(h) : l^2(\Gamma)^m \longrightarrow l^2(\Gamma)$  is surjective. Let  $\{E_\lambda \mid \lambda \geq 0\}$  be the spectral family of the positive operator  $h \circ h^*$ . Then  $\{i(E_\lambda) \mid \lambda \geq 0\}$  is the spectral family of the positive operator  $i(h) \circ i(h)^*$ . Notice that  $h$  resp.  $i(h)$  is surjective if and only if  $E_\lambda = 0$  resp.  $i(E_\lambda) = 0$  for some  $\lambda > 0$ . Because  $E_\lambda = 0$  resp.  $i(E_\lambda) = 0$  is equivalent to  $\dim_{\mathcal{N}(\Delta)}(\text{im}(\nu^{-1}(E_\lambda))) = 0$  resp.  $\dim_{\mathcal{N}(\Gamma)}(\text{im}(\nu^{-1}(i(E_\lambda)))) = 0$  and  $\text{im}(\nu^{-1}(i(E_\lambda))) = i_*\text{im}(\nu^{-1}(E_\lambda))$ , the claim follows from Step 6. This finishes the proof of Theorem 3.3. ■

The proof of the first two assertions of Theorem 3.3 would be obvious if we would know that  $\mathcal{N}(\Gamma)$  viewed as a  $\mathcal{N}(\Delta)$ -module is projective. Notice that this is a stronger statement than proven in Theorem 3.3. One would have to show that the higher Ext-groups instead of the Tor-groups vanish to get this stronger statement. However, the proof for the Tor-groups does not go through directly since the Ext-groups are not compatible with colimits.

**Lemma 3.4** *Let  $H \subset \Gamma$  be a subgroup. Then*

1.  $\dim(\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}[\Gamma/H]) = |H|^{-1}$ , where  $|H|^{-1}$  is defined to be zero if  $H$  is infinite;

2.  $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}[\Gamma/H]$  is trivial if and only if  $H$  is non-amenable;

3. If  $\Gamma$  is infinite and  $V$  is a  $\mathbb{C}\Gamma$ -module which is finite-dimensional over  $\mathbb{C}$ , then

$$\dim(\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} V) = 0.$$

**Proof :** 3.) Since  $V$  is finitely generated as  $\mathbb{C}\Gamma$ -module  $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} V$  is a finitely generated  $\mathcal{N}(\Gamma)$ -module. Because of Theorem 0.6 it suffices to show that there is no  $\mathcal{N}(\Gamma)$ -homomorphism from  $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} V$  to  $\mathcal{N}(\Gamma)$ . This is equivalent to the claim that there is no  $\mathbb{C}\Gamma$ -homomorphism from  $V$  to  $\mathcal{N}(\Gamma)$ . Since the map  $\mathcal{N}(\Gamma) \longrightarrow l^2(\Gamma)$  given by evaluation at the unit element  $e \in \Gamma \subset l^2(\Gamma)$  is  $\Gamma$ -equivariant and injective it suffices to show that  $l^2(\Gamma)$  contains no  $\Gamma$ -invariant linear subspace  $W$  which is finite-dimensional as complex vector space. Since any finite-dimensional topological vector space is complete,  $W$  is a Hilbert  $\mathcal{N}(\Gamma)$ -submodule. Let  $\text{pr} : l^2(\Gamma) \longrightarrow l^2(\Gamma)$  be an orthogonal  $\Gamma$ -equivariant projection onto  $W$ . Then we get for any  $\gamma \in \Gamma$

$$\dim(W) = \langle \text{pr}(\gamma), \gamma \rangle. \quad (3.5)$$

Let  $\{v_1, v_2, \dots, v_r\}$  be an orthonormal basis for the Hilbert subspace  $W \subset l^2(\Gamma)$ . For  $\gamma \in \Gamma$  we write  $\text{pr}(\gamma) = \sum_{i=1}^r \lambda_i(\gamma) \cdot v_i$ . We get from  $\|\text{pr}(\gamma)\|^2 \leq 1$

$$|\lambda_i(\gamma)| \leq 1. \quad (3.6)$$

Given  $\epsilon > 0$ , we can choose  $\gamma(\epsilon)$  satisfying

$$\langle v_i, \gamma(\epsilon) \rangle \leq r^{-1} \cdot \epsilon \quad \text{for } i = 1, 2, \dots, r. \quad (3.7)$$

Now 3.6 and 3.7 imply

$$\langle \text{pr}(\gamma(\epsilon)), \gamma(\epsilon) \rangle \leq \epsilon. \quad (3.8)$$

Since 3.8 holds for all  $\epsilon > 0$ , we conclude  $\dim(W) = 0$  and hence  $W = 0$  from equation 3.5.

1. and 2.) If  $i : H \longrightarrow \Gamma$  is the inclusion, then  $i_*(\mathcal{N}(H) \otimes_{\mathbb{C}H} \mathbb{C})$  and  $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}[\Gamma/H]$  are isomorphic as  $\mathcal{N}(\Gamma)$ -modules. Because of Theorem 3.3 it remains to treat the special case  $\Gamma = H$  for the first two assertions. The first assertion follows from the third for infinite  $\Gamma$  and is obvious for finite  $\Gamma$ . Next we prove the second assertion.

Let  $S$  be a set of generators of  $\Gamma$ . Then  $\oplus_{s \in S} \mathbb{Z}\Gamma \xrightarrow{\oplus_{s \in S} r_{s-1}} \mathbb{C}\Gamma \xrightarrow{\epsilon} \mathbb{C} \longrightarrow 0$  is exact where  $\epsilon(\sum_{\gamma \in \Gamma} \lambda_\gamma \cdot \gamma) = \sum_{\gamma \in \Gamma} \lambda_\gamma$ . Since the tensor product is right exact, we obtain an exact sequence  $\oplus_{s \in S} \mathcal{N}(\Gamma) \xrightarrow{\oplus_{s \in S} r_{s-1}} \mathcal{N}(\Gamma) \xrightarrow{\epsilon} \mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C} \longrightarrow 0$ . Hence  $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}$  is trivial if and only if  $\oplus_{s \in S} \mathcal{N}(\Gamma) \xrightarrow{\oplus_{s \in S} r_{s-1}} \mathcal{N}(\Gamma)$  is surjective. This is equivalent to the existence of a finite subset  $T \subset S$  such that  $\oplus_{s \in T} \mathcal{N}(\Gamma) \xrightarrow{\oplus_{s \in T} r_{s-1}} \mathcal{N}(\Gamma)$  is surjective. Let  $\Delta \subset \Gamma$  be the subgroup generated by  $T$ . Then the map above is the induction with the inclusion of  $\Delta$  in  $\Gamma$  applied to  $\oplus_{t \in T} \mathcal{N}(\Delta) \xrightarrow{\oplus_{t \in T} r_{t-1}} \mathcal{N}(\Delta)$ . Hence we conclude from Theorem 3.3.1 that  $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}$  is trivial if  $\mathcal{N}(\Delta) \otimes_{\mathbb{C}\Delta} \mathbb{C}$  is trivial for some finitely generated subgroup  $\Delta \subset \Gamma$ . Since  $\Gamma$  is amenable if and only if each of its finitely generated subgroups is amenable [42, Proposition 0.16 on page 14], we can assume without loss of generality that  $\Gamma$  is finitely



generated, i.e.  $S$  is finite. We can also assume that  $S$  is symmetric, i.e.  $s \in S$  implies  $s^{-1} \in S$ .

Because the functor  $\nu$  of Theorem 1.8 is exact,  $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}$  is trivial if and only if the operator  $f : \oplus_{s \in S} l^2(\Gamma) \xrightarrow{\oplus_{s \in S} r_{s^{-1}}} l^2(\Gamma)$  is surjective. This is equivalent to the bijectivity of the operator

$$\frac{1}{2 \cdot |S|} f \circ f^* : l^2(\Gamma) \xrightarrow{\text{id} - \sum_{s \in S} \frac{1}{|S|} r_s} l^2(\Gamma).$$

It is bijective if and only if the spectral radius of the operator  $l^2(\Gamma) \xrightarrow{\sum_{s \in S} \frac{1}{|S|} r_s} l^2(\Gamma)$  is different from 1. Since this operator is convolution with a probability distribution whose support contains  $S$ , namely

$$P : \Gamma \longrightarrow [0, 1] \quad \gamma \mapsto \begin{cases} |S|^{-1} & \gamma \in S \\ 0 & \gamma \notin S \end{cases}$$

the spectral radius is 1 precisely if  $\Gamma$  is amenable [25]. This finishes the proof of Lemma 3.4.  $\blacksquare$

## 4. $L^2$ -invariants for arbitrary $\Gamma$ -spaces

In this section we extend the notion of  $L^2$ -Betti numbers for regular coverings of  $CW$ -complexes of finite type (i.e. with finite skeletons) with  $\Gamma$  as group of deck transformations to (compactly generated) topological spaces with action of a (discrete) group  $\Gamma$ . We will continue with using Notation 1.1.

**Definition 4.1** *Let  $X$  be a (left)  $\Gamma$ -space and  $V$  be a  $\mathcal{A}$ - $\mathbb{Z}\Gamma$ -bimodule. Let  $H_p^\Gamma(X; V)$  be the singular homology of  $X$  with coefficients in  $V$ , i.e. the  $\mathcal{A}$ -module given by the homology of the  $\mathcal{A}$ -chain complex  $V \otimes_{\mathbb{Z}\Gamma} C_*^{\text{sing}}(X)$ , where  $C_*^{\text{sing}}(X)$  denotes the singular  $\mathbb{Z}\Gamma$ -chain complex of  $X$ . Define the  $p$ -th  $L^2$ -Betti number of  $X$  with coefficients in  $V$  by*

$$b_p^{(2)}(X; V) := \dim_{\mathcal{A}} (H_p^\Gamma(X; V)) \in [0, \infty].$$

and the  $p$ -th  $L^2$ -Betti number of the group  $\Gamma$  by

$$b_p^{(2)}(\Gamma) := b_p^{(2)}(E\Gamma; \mathcal{N}(\Gamma)). \quad \blacksquare$$

Next we compare cellular and singular chain complexes and show that it does not matter whether we use singular or cellular chain complexes in the case that  $X$  is a  $\Gamma$ - $CW$ -complex. For basic definitions and facts about  $\Gamma$ - $CW$ -complexes we refer for instance to [14, sections II.1 and II.2], [30, sections 1 and 2].

**Lemma 4.2** *Let  $X$  be a  $\Gamma$ - $CW$ -complex. Then there is a up to  $\mathbb{Z}\Gamma$ -homotopy unique and in  $X$  natural  $\mathbb{Z}\Gamma$ -chain homotopy equivalence*

$$f(X) : C_*^{\text{cell}}(X) \longrightarrow C_*^{\text{sing}}(X).$$

In particular we get for any  $\mathcal{A}$ - $\mathbb{Z}\Gamma$ -bimodule  $V$  an in  $X$  and  $V$  natural isomorphism

$$H_p(V \otimes_{\mathbb{Z}\Gamma} C_*^{\text{cell}}(X)) \xrightarrow{\cong} H_p(V \otimes_{\mathbb{Z}\Gamma} C_*^{\text{sing}}(X)). \quad \blacksquare$$

**Proof :** Obviously the second assertion follows from the first assertion which is proven as follows.

Let  $Y$  be a  $CW$ -complex with cellular  $\mathbb{Z}$ -chain complex  $C_*^{\text{cell}}$  and singular  $\mathbb{Z}$ -chain complex  $C_*^{\text{sing}}$ . We define a third (intermediate)  $\mathbb{Z}$ -chain complex  $C_*^{\text{inte}}(Y)$  as the subcomplex of  $C_*^{\text{sing}}$  whose  $n$ -th chain module is the kernel of

$$C_n^{\text{sing}}(Y_n) \xrightarrow{c_n^{\text{sing}}} C_{n-1}^{\text{sing}}(Y_n) \longrightarrow C_{n-1}^{\text{sing}}(Y_n, Y_{n-1}).$$

There are an in  $Y$  natural inclusion and an in  $Y$  natural epimorphism of  $\mathbb{Z}$ -chain complexes

$$i(Y) : C_*^{\text{inte}}(Y) \longrightarrow C_*^{\text{sing}}(Y); \quad (4.3)$$

$$p(Y) : C_*^{\text{inte}}(Y) \longrightarrow C_*^{\text{cell}}(Y); \quad (4.4)$$

which induce isomorphisms on homology [30, page 263].

If  $\Gamma$  acts freely on the  $\Gamma$ - $CW$ -complex  $X$ , then  $C_*^{\text{cell}}$  and  $C_*^{\text{sing}}$  are free  $\mathbb{Z}\Gamma$ -chain complexes and we get a  $\mathbb{Z}\Gamma$ -chain homotopy equivalence well-defined up to  $\mathbb{Z}\Gamma$ -homotopy from the fundamental theorem of homological algebra and the fact that the chain maps 4.3 and 4.4 induce isomorphisms on homology. In the general case one has to go to the orbit category  $\text{Or}(\Gamma)$  and apply module theory over this category instead of over  $\mathbb{Z}\Gamma$ .

The orbit category  $\text{Or}(\Gamma)$  has as objects homogenous spaces and as morphisms  $\Gamma$ -maps. The  $\Gamma$ - $CW$ -complex  $X$  defines a contravariant functor

$$\underline{X} : \text{Or}(\Gamma) \longrightarrow \{CW - \text{COMPLEXES}\} \quad \Gamma/H \mapsto \text{map}(\Gamma/H, X)^\Gamma = X^H.$$

Its composition with the functor  $C_*^{\text{cell}}$ ,  $C_*^{\text{inte}}$  resp.  $C_*^{\text{sing}}$  from the category of  $CW$ -complexes to the category of chain complexes yields  $\mathbb{Z}\text{Or}(\Gamma)$ -chain complexes, i.e. contravariant functors

$$\begin{aligned} C_*^{\text{cell}}(\underline{X}) : \text{Or}(\Gamma) &\longrightarrow \{\mathbb{Z} - \text{CHAIN-COMPLEXES}\}; \\ C_*^{\text{inte}}(\underline{X}) : \text{Or}(\Gamma) &\longrightarrow \{\mathbb{Z} - \text{CHAIN-COMPLEXES}\}; \\ C_*^{\text{sing}}(\underline{X}) : \text{Or}(\Gamma) &\longrightarrow \{\mathbb{Z} - \text{CHAIN-COMPLEXES}\}. \end{aligned}$$

We obtain natural transformations from the natural chain maps 4.3 and 4.4

$$i(\underline{X}) : C_*^{\text{inte}}(\underline{X}) \longrightarrow C_*^{\text{sing}}(\underline{X}); \quad (4.5)$$

$$p(\underline{X}) : C_*^{\text{inte}}(\underline{X}) \longrightarrow C_*^{\text{cell}}(\underline{X}); \quad (4.6)$$

which induce isomorphisms on homology. Hence it suffices to show that  $C_*^{\text{cell}}(\underline{X})$  and  $C_*^{\text{sing}}(\underline{X})$  are free and hence projective in the sense of [30, Definition 9.17] because then we obtain a homotopy equivalence of  $\mathbb{Z}\text{Or}(\Gamma)$ -chain complexes from  $C_*^{\text{cell}}(\underline{X})$  to  $C_*^{\text{sing}}(\underline{X})$  [30, Lemma 11.3] whose evaluation at  $\Gamma/1$  is the desired  $\mathbb{Z}\Gamma$ -chain homotopy equivalence. The proofs that these two chain complexes are free are simple versions of the arguments in [30, Lemma 13.2]. Notice that in [30] the  $\Gamma$ - $CW$ -complex is required to be proper, but this condition is needed there only because there  $\Gamma$  is assumed to be a Lie group and universal coverings are built in, and can be dropped in the discrete case. ■

**Remark 4.7** Originally the  $L^2$ -Betti numbers of a regular covering  $\overline{M} \longrightarrow M$  of a closed Riemannian manifold with group of deck transformations  $\Gamma$  were defined by Atiyah [2] in

terms of the heat kernel as explained in 0.2 in the introduction. It follows from the  $L^2$ -Hodge-deRham theorem [16] that this analytic definition agrees with the combinatorial definition of  $b_p^{(2)}(\overline{X})$  in terms of the associated cellular  $L^2$ -chain complex and the von Neumann dimension of finitely generated Hilbert  $\mathcal{N}(\Gamma)$ -modules for a triangulation  $X$  of  $M$ . Because of Lemma 4.2 this combinatorial definition agrees with the Definition 4.1.

Analogously to the case of  $L^2$ -Betti numbers we will extend the notion of Novikov-Shubin invariants for regular coverings of compact Riemannian manifolds to arbitrary  $\Gamma$ -spaces and prove that they are positive for the universal covering of a aspherical closed manifold with elementary-amenable fundamental group in another paper. ■

The next results are well-known in the case where  $X$  is a regular covering of a  $CW$ -complex of finite type. We call a map  $g : Y \longrightarrow Z$  *homologically  $n$ -connected* for  $n \geq 1$  if the map induced on singular homology with complex coefficients  $g_* : H_k^{\text{sing}}(Y; \mathbb{C}) \longrightarrow H_k^{\text{sing}}(Z; \mathbb{C})$  is bijective for  $k < n$  and surjective for  $k = n$ . The map  $g$  is called a *weak homology equivalence* if it is  $n$ -connected for all  $n \geq 1$ .

**Lemma 4.8** *Let  $f : X \longrightarrow Y$  be a  $\Gamma$ -map and let  $V$  be a  $\mathcal{A}$ - $\mathbb{Z}\Gamma$ -bimodule.*

1. *Suppose for  $n \geq 1$  that for each subgroup  $H \subset \Gamma$  the induced map  $f^H : X^H \longrightarrow Y^H$  is homologically  $n$ -connected. Then the map induced by  $f$*

$$f_* : H_p^\Gamma(X; V) \longrightarrow H_p^\Gamma(Y; V)$$

*is bijective for  $p < n$  and surjective for  $p = n$  and we get*

$$\begin{aligned} b_p^{(2)}(X; V) &= b_p^{(2)}(Y; V) && \text{for } p < n; \\ b_p^{(2)}(X; V) &\geq b_p^{(2)}(Y; V) && \text{for } p = n; \end{aligned}$$

2. *Suppose such that for each subgroup  $H \subset \Gamma$  the induced map  $f^H : X^H \longrightarrow Y^H$  is a weak homology equivalence. Then for all  $p \geq 0$  the map induced by  $f$*

$$f_* : H_p^\Gamma(X; V) \longrightarrow H_p^\Gamma(Y; V)$$

*is bijective and we get*

$$b_p^{(2)}(X; V) = b_p^{(2)}(Y; V). \quad \blacksquare$$

**Proof :** We give only the proof of the second assertion, the one of the first assertion is an elementary modification. The map  $f$  induces a homotopy equivalence of  $\mathbb{Z}\text{Or}(\Gamma)$ -chain complexes  $C_*^{\text{sing}}(f) : C_*^{\text{sing}}(\underline{X}) \longrightarrow C_*^{\text{sing}}(\underline{Y})$  in the notation of the proof of Lemma 4.2 since the singular  $\mathbb{Z}\text{Or}(\Gamma)$ -chain complexes of  $\underline{X}$  and  $\underline{Y}$  are free in the sense of [30, Definition 9.17](see [30, Lemma 11.3]). Its evaluation at  $\Gamma/1$  is a  $\mathbb{Z}\Gamma$ -chain equivalence. Hence  $f$  induces a chain equivalence

$$V \otimes_{\mathbb{Z}\Gamma} C_*^{\text{sing}}(f) : V \otimes_{\mathbb{Z}\Gamma} C_*^{\text{sing}}(X) \longrightarrow V \otimes_{\mathbb{Z}\Gamma} C_*^{\text{sing}}(Y)$$

and Lemma 4.8 follows. ■

We get as a direct consequence from Theorem 3.3

**Theorem 4.9** *Let  $i : \Delta \longrightarrow \Gamma$  be an inclusion of groups and let  $X$  be a  $\Delta$ -space. Then*

$$\begin{aligned} H_p^\Gamma(\Gamma \times_\Delta X; \mathcal{N}(\Gamma)) &= i_* H_p^\Delta(X; \mathcal{N}(\Delta)); \\ b_p^{(2)}(\Gamma \times_\Delta X; \mathcal{N}(\Gamma)) &= b_p^{(2)}(X; \mathcal{N}(\Delta)). \quad \blacksquare \end{aligned}$$

**Theorem 4.10** *Let  $X$  be a path-connected  $\Gamma$ -space. Then*

1. *There is an isomorphism of  $\mathcal{N}(\Gamma)$ -modules  $H_0^\Gamma(X; \mathcal{N}(\Gamma)) \cong \mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} \mathbb{C}$ ;*
2.  *$b_0^{(2)}(X; \mathcal{N}(\Gamma)) = |\Gamma|^{-1}$ , where  $|\Gamma|^{-1}$  is defined to be zero if the order  $|\Gamma|$  of  $\Gamma$  is infinite;*
3.  *$H_0^\Gamma(X; \mathcal{N}(\Gamma))$  is trivial if and only if  $\Gamma$  is non-amenable.*

**Proof :** The first assertion follows from the fact that  $C_1^{\text{sing}}(X) \longrightarrow C_0^{\text{sing}}(X) \longrightarrow \mathbb{Z} \longrightarrow 0$  is an exact sequence of  $\mathbb{Z}\Gamma$ -modules and the tensor product is right exact. The other two assertions follow from Lemma 3.4.  $\blacksquare$

**Remark 4.11** Let  $\widetilde{M} \longrightarrow M$  be the universal covering of a closed Riemannian manifold with fundamental group  $\pi$ . Brooks [7] has shown that the analytic Laplace operator  $\Delta_0$  on  $\widetilde{M}$  in dimension zero has zero not in its spectrum if and only if  $\pi$  is non-amenable. Now  $\Delta_0$  has zero not in its spectrum if and only if  $H_0^\pi(\widetilde{M}, \mathcal{N}(\pi))$  is trivial because of [34, paragraph after Definition 3.11, Theorem 6.1] and the fact that the analytic and combinatorial spectral density function are dilatationally equivalent [19]. Hence Theorem 4.10 generalizes the result of Brooks. Notice that both Brooks' and our proof use [25]. Compare also with the result [23, Corollary III.2.4 on page 188] that a group  $\Gamma$  is non-amenable if and only if  $H^1(\Gamma, l^2(\Gamma))$  is Hausdorff.  $\blacksquare$

**Remark 4.12** Next we compare our approach with the one in [11, section 2]. We begin with the case of a countable simplicial complex  $X$  with free simplicial  $\Gamma$ -action. Then for any exhaustion  $X_0 \subset X_1 \subset X_2 \subset \dots \subset X$  by  $\Gamma$ -equivariant simplicial subcomplexes for which  $X/\Gamma$  is compact, the  $p$ -th  $L^2$ -Betti number in the sense and notation of [11, 2.8 on page 198] is given by

$$b_p^{(2)}(X : \Gamma) = \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \dim_{\mathcal{N}(\Gamma)} \left( \text{im}(\overline{H}_{(2)}^p(X_k : \Gamma) \xrightarrow{i_{j,k}^*} \overline{H}_{(2)}^p(X_j : \Gamma)) \right),$$

where  $i_{j,k} : X_j \longrightarrow X_k$  is the inclusion for  $j \leq k$ . We get from [34, Lemma 1.3] and Lemma 4.2

$$\begin{aligned} & \dim_{\mathcal{N}(\Gamma)} \left( \text{im}(\overline{H}_{(2)}^p(X_k : \Gamma) \xrightarrow{i_{j,k}^*} \overline{H}_{(2)}^p(X_j : \Gamma)) \right) \\ &= \dim_{\mathcal{N}(\Gamma)} \left( \text{im}(H_p^\Gamma(X_j; \mathcal{N}(\Gamma)) \xrightarrow{(i_{j,k})_*} H_p^\Gamma(X_k; \mathcal{N}(\Gamma))) \right). \end{aligned}$$

Hence we conclude from Theorem 2.9 that the definitions in [11, 2.8 on page 198] and in 4.1 agree:

$$b_p^{(2)}(X : \Gamma) = b_p^{(2)}(X; \mathcal{N}(\Gamma)). \quad (4.13)$$

If  $\Gamma$  is countable and  $X$  is a countable simplicial complex with simplicial  $\Gamma$ -action, then by [11, Proposition 2.2 on page 198] and by 4.13

$$b_p^{(2)}(X : \Gamma) = b_p^{(2)}(E\Gamma \times X : \Gamma); \quad (4.14)$$

$$b_p^{(2)}(X : \Gamma) = b_p^{(2)}(E\Gamma \times X; \mathcal{N}(\Gamma)). \quad (4.15)$$

Cheeger and Gromov [11, Section 2] define  $L^2$ -cohomology and  $L^2$ -Betti numbers of a  $\Gamma$ -space  $X$  by considering the category whose objects are  $\Gamma$ -maps  $f : Y \longrightarrow X$  for a simplicial complex  $Y$  with cocompact free simplicial  $\Gamma$ -action and then using inverse limits to extend the classical notions for finite free  $\Gamma$ -CW-complexes such as  $Y$  to  $X$ . Our approach avoids the technical difficulties concerning inverse limits and is closer to standard notions, the only non-standard part is the verification of the properties of the extended dimension function (Theorem 0.6 and Theorem 3.3).

## 5. Amenable groups

In this section we investigate amenable groups. For information about amenable groups we refer for instance to [42]. The main technical result of this section is the next lemma whose proof uses ideas of the proof of [11, Lemma 3.1 on page 203].

**Theorem 5.1** *Let  $\Gamma$  be amenable and  $M$  be a  $\mathbb{C}\Gamma$ -module. Then*

$$\dim_{\mathcal{N}(\Gamma)} (\text{Tor}_p^{\mathbb{C}\Gamma}(M, \mathcal{N}(\Gamma))) = 0 \quad \text{for } p \geq 1,$$

where we consider  $\mathcal{N}(\Gamma)$  as a  $\mathcal{N}(\Gamma)$ - $\mathbb{C}\Gamma$ -bimodule.

**Proof**: Step 1: If  $M$  is a finitely presented  $\mathbb{C}\Gamma$ -module, then  $\dim_{\mathcal{N}(\Gamma)} (\text{Tor}_1^{\mathbb{C}\Gamma}(M, \mathcal{N}(\Gamma))) = 0$ .

Choose a finite presentation

$$\oplus_{i=1}^m \mathbb{C}\Gamma \xrightarrow{f} \oplus_{i=1}^n \mathbb{C}\Gamma \xrightarrow{p} M \longrightarrow 0.$$

For an element  $u = \sum_{\gamma} \lambda_{\gamma} \cdot \gamma$  in  $l^2(\Gamma)$  define its *support*

$$\text{supp}(u) := \{\gamma \in \Gamma \mid \lambda_{\gamma} \neq 0\} \subset \Gamma.$$

Let  $B \in M(m, n, \mathbb{C}\Gamma)$  be the matrix describing  $f$ , i.e. the component  $f_{i,j} : \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma$  is given by right multiplication with  $b_{i,j}$ . Define the finite subset  $S$  by

$$S := \{\gamma \mid \gamma \text{ or } \gamma^{-1} \in \bigcup_{i,j} \text{supp}(b_{i,j})\}.$$

Let  $f^{(2)} : \oplus_{i=1}^m l^2(\Gamma) \longrightarrow \oplus_{i=1}^n l^2(\Gamma)$  be the bounded  $\Gamma$ -equivariant operator induced by  $f$ . Denote by  $K$  the  $\Gamma$ -invariant linear subspace of  $\oplus_{i=1}^m l^2(\Gamma)$  which is the image of the kernel of  $f$  under the canonical inclusion  $k : \oplus_{i=1}^m \mathbb{C}\Gamma \longrightarrow \oplus_{i=1}^m l^2(\Gamma)$ . Next we show for the closure  $\overline{K}$  of  $K$

$$\overline{K} = \ker(f^{(2)}). \quad (5.2)$$

Let  $\text{pr} : \oplus_{i=1}^m l^2(\Gamma) \longrightarrow \oplus_{i=1}^m l^2(\Gamma)$  be the orthogonal projection onto the closed  $\Gamma$ -invariant subspace  $\overline{K}^\perp \cap \ker(f^{(2)})$ . The von Neumann dimension of  $\text{im}(\text{pr})$  is zero if and only if  $\text{pr}$  itself is zero. Hence 5.2 will follow if we can prove

$$\text{tr}_{\mathcal{N}(\Gamma)}(\text{pr}) = 0. \quad (5.3)$$

Let  $\epsilon > 0$  be given. Since  $\Gamma$  is amenable, there is a finite non-empty subset  $A \subset \Gamma$  satisfying [5, Theorem F.6.8 on page 308]

$$\frac{|\partial_S A|}{|A|} \leq \epsilon, \quad (5.4)$$

where  $\partial_S A$  is defined by  $\{a \in A \mid \text{there is } s \in S \text{ with } as \notin A\}$ . Define

$$\Delta := \{\gamma \in \Gamma \mid \gamma \in \partial_S A \text{ or } \gamma s \in \partial_S A \text{ for some } s \in S\} = \partial_S A \cup (\cup_{s \in S} (\partial_S A)s).$$

Let  $\text{pr}_A : l^2(\Gamma) \longrightarrow l^2(\Gamma)$  be the projection sending  $\sum_{\gamma \in \Gamma} \lambda_\gamma \cdot \gamma$  to  $\sum_{\gamma \in A} \lambda_\gamma \cdot \gamma$ . Define  $\text{pr}_\Delta : l^2(\Gamma) \longrightarrow l^2(\Gamma)$  analogously. Next we show for  $s \in S$  and  $u \in l^2(\Gamma)$

$$\text{pr}_A \circ r_s(u) = r_s \circ \text{pr}_A(u) \quad , \text{if } \text{pr}_\Delta(u) = 0, \quad (5.5)$$

where  $r_s : l^2(\Gamma) \longrightarrow l^2(\Gamma)$  is right multiplication with  $s$ . Since  $s \in S$  implies  $s^{-1} \in S$ , we get the following equality of subsets of  $\Gamma$

$$\{\gamma \in \Gamma \mid \gamma s \in A, \gamma \notin \Delta\} = \{\gamma \in \Gamma \mid \gamma \in A, \gamma \notin \Delta\}.$$

Now 5.5 follows from the following calculation for  $u = \sum_{\gamma \in \Gamma, \gamma \notin \Delta} \lambda_\gamma \cdot \gamma \in l^2(\Gamma)$

$$\begin{aligned} \text{pr}_A \circ r_s(u) &= \sum_{\gamma s \in A, \gamma \notin \Delta} \lambda_\gamma \cdot \gamma s \\ &= \sum_{\gamma \in A, \gamma \notin \Delta} \lambda_\gamma \cdot \gamma s \\ &= \left( \sum_{\gamma \in A, \gamma \notin \Delta} \lambda_\gamma \cdot \gamma \right) \cdot s \\ &= r_s \circ \text{pr}_A(u). \end{aligned}$$

We have defined  $S$  such that each entry in the matrix  $B$  describing  $f$  is a linear combination of elements in  $S$ . Hence 5.5 implies

$$(\oplus_{j=1}^n \text{pr}_A) \circ f^{(2)}(u) = f^{(2)} \circ (\oplus_{i=1}^m \text{pr}_A)(u) \quad , \text{if } \text{pr}_\Delta(u_i) = 0 \text{ for } i = 1, 2, \dots, m.$$

Notice that the image of  $\oplus_{i=1}^m \text{pr}_A$  lies in  $\oplus_{i=1}^m \mathbb{C}\Gamma$ . We conclude

$$\oplus_{i=1}^m \text{pr}_A(u) \in K \quad , \text{ if } u \in \ker(f^{(2)}), \text{pr}_\Delta(u_i) = 0 \text{ for } i = 1, 2, \dots, m.$$

This shows

$$(\text{pr} \circ \oplus_{i=1}^m \text{pr}_A) (\ker(f^{(2)}) \cap \oplus_{i=1}^m \ker(\text{pr}_\Delta)) = 0.$$

Since  $\ker(\text{pr}_\Delta)$  has complex codimension  $|\Delta|$  in  $l^2(\Gamma)$  and  $|\Delta| \leq (|S| + 1) \cdot |\partial_S A|$ , we conclude for the complex dimension  $\dim_{\mathbb{C}}$  of complex vector spaces

$$\dim_{\mathbb{C}} ((\text{pr} \circ \oplus_{i=1}^m \text{pr}_A) (\ker(f^{(2)}))) \leq m \cdot (|S| + 1) \cdot |\partial_S A|. \quad (5.6)$$

Since  $\text{pr} \circ \text{pr}_A$  is an endomorphism of Hilbert spaces with finite-dimensional image, it is trace-class and its trace  $\text{tr}_{\mathbb{C}}(\text{pr} \circ \text{pr}_A)$  is defined. We get

$$\text{tr}_{\mathcal{N}(\Gamma)}(\text{pr}) \leq \frac{\text{tr}_{\mathbb{C}}(\text{pr} \circ \oplus_{i=1}^m \text{pr}_A)}{|A|} \quad (5.7)$$

from the following computation for  $e \in \Gamma \subset l^2(\Gamma)$  the unit element

$$\begin{aligned} \text{tr}_{\mathcal{N}(\Gamma)}(\text{pr}) &= \sum_{i=1}^m \langle \text{pr}_{i,i}(e), e \rangle \\ &= \frac{1}{|A|} \sum_{i=1}^m |A| \cdot \langle \text{pr}_{i,i}(e), e \rangle \\ &= \frac{1}{|A|} \sum_{i=1}^m \sum_{\gamma \in A} \langle \text{pr}_{i,i}(\gamma), \gamma \rangle \\ &= \frac{1}{|A|} \sum_{i=1}^m \sum_{\gamma \in A} \langle \text{pr}_{i,i} \circ \text{pr}_A(\gamma), \gamma \rangle \\ &= \frac{1}{|A|} \sum_{i=1}^m \sum_{\gamma \in \Gamma} \langle \text{pr}_{i,i} \circ \text{pr}_A(\gamma), \gamma \rangle \\ &= \frac{1}{|A|} \sum_{i=1}^m \text{tr}_{\mathbb{C}}(\text{pr}_{i,i} \circ \text{pr}_A) \\ &= \frac{1}{|A|} \text{tr}_{\mathbb{C}}(\text{pr} \circ (\oplus_{i=1}^m \text{pr}_A)). \end{aligned}$$

If  $H$  is a Hilbert space and  $f : H \rightarrow H$  is a bounded operator with finite-dimensional image, then  $\text{tr}_{\mathbb{C}}(f) \leq \|f\| \cdot \dim_{\mathbb{C}}(f(\text{im}(f)))$ . Since the image of  $\text{pr}$  is contained in  $\ker(f^{(2)})$  and  $\text{pr}$  and  $\text{pr}_A$  have operator norm 1, we conclude

$$\text{tr}_{\mathbb{C}}(\text{pr} \circ \oplus_{i=1}^m \text{pr}_A) \leq \dim_{\mathbb{C}} ((\text{pr} \circ \oplus_{i=1}^m \text{pr}_A) (\ker(f^{(2)}))). \quad (5.8)$$

Equations 5.4, 5.6, 5.7 and 5.8 imply

$$\text{tr}_{\mathcal{N}(\Gamma)}(\text{pr}) \leq m \cdot (|S| + 1) \cdot \epsilon.$$

Since this holds for all  $\epsilon > 0$ , we get 5.3 and hence 5.2 is true.

Let  $\text{pr}_{\overline{K}} : \oplus_{i=1}^m l^2(\Gamma) \longrightarrow \oplus_{i=1}^m l^2(\Gamma)$  be the projection onto  $\overline{K}$ . Let  $i : \ker(f) \longrightarrow \oplus_{i=1}^m \mathbb{C}\Gamma$  be the inclusion. It induces a map

$$j : \mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \ker(f) \xrightarrow{\text{id} \otimes_{\mathbb{C}\Gamma} i} \mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \oplus_{i=1}^m \mathbb{C}\Gamma \xrightarrow{\cong} \oplus_{i=1}^m \mathcal{N}(\Gamma).$$

Next we want to show

$$\text{im}(\nu^{-1}(\text{pr}_{\overline{K}})) = \overline{\text{im}(j)}. \quad (5.9)$$

Let  $x \in \ker(f)$ . Then

$$(\text{id} - \nu^{-1}(\text{pr}_{\overline{K}})) \circ j(1 \otimes x) = (\text{id} - \text{pr}_{\overline{K}}) \circ k \circ i(x), \quad (5.10)$$

where  $k : \oplus_{i=1}^m \mathbb{C}\Gamma \longrightarrow \oplus_{i=1}^m l^2(\Gamma)$  is the inclusion. Since  $(\text{id} - \text{pr}_{\overline{K}})$  is trivial on  $K$  we get  $(\text{id} - \text{pr}_{\overline{K}}) \circ k \circ i = 0$ . Now we conclude from 5.10 that  $\text{im}(j) \subset \ker(\text{id} - \nu^{-1}(\text{pr}_{\overline{K}}))$  and hence  $\text{im}(j) \subset \text{im}(\nu^{-1}(\text{pr}_{\overline{K}}))$  holds. This shows  $\overline{\text{im}(j)} \subset \text{im}(\nu^{-1}(\text{pr}_{\overline{K}}))$ . It remains to prove for any  $\mathcal{N}(\Gamma)$ -map  $g : \oplus_{i=1}^m \mathcal{N}(\Gamma) \longrightarrow \oplus_{i=1}^m \mathcal{N}(\Gamma)$  with  $\text{im}(j) \subset \ker(g)$  that  $g \circ \nu^{-1}(\text{pr}_{\overline{K}})$  is trivial. Obviously  $K \subset \ker(\nu(g))$ . Since  $\ker(\nu(g))$  is a closed subspace, we get  $\overline{K} \subset \ker(\nu(g))$ . We conclude  $\nu(g) \circ \text{pr}_{\overline{K}} = 0$  and hence  $g \circ \nu^{-1}(\text{pr}_{\overline{K}}) = 0$ . This finishes the proof of 5.9.

Since  $\nu^{-1}$  preserves exactness by Theorem 1.8 and  $\text{id} \otimes_{\mathbb{C}\Gamma} f = \nu^{-1}(f^{(2)})$ , we conclude from 5.2 and 5.9 that the sequence

$$\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \ker(f) \xrightarrow{\text{id} \otimes_{\mathbb{C}\Gamma} i} \mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \oplus_{i=1}^m \mathbb{C}\Gamma \xrightarrow{\text{id} \otimes_{\mathbb{C}\Gamma} f} \mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} \oplus_{i=1}^m \mathbb{C}\Gamma$$

is weakly exact. Continuity of the dimension function (see Theorem 0.6.4) implies

$$\dim_{\mathcal{N}(\Gamma)} (\ker(\text{id} \otimes_{\mathbb{C}\Gamma} f) / \text{im}(\text{id} \otimes_{\mathbb{C}\Gamma} i)) = 0.$$

Since  $\text{Tor}_1^{\mathbb{C}\Gamma}(M; \mathcal{N}(\Gamma)) = \ker(\text{id} \otimes_{\mathbb{C}\Gamma} f) / \text{im}(\text{id} \otimes_{\mathbb{C}\Gamma} i)$  holds, Step 1 follows.

Step 2: If  $M$  is a  $\mathbb{C}\Gamma$ -module, then  $\dim_{\mathcal{N}(\Gamma)} (\text{Tor}_1^{\mathbb{C}\Gamma}(M, \mathcal{N}(\Gamma))) = 0$ .

Obviously  $M$  is the union of its finitely generated submodules. Any finitely generated module  $M$  is a colimit over a directed system of finitely presented modules, namely, choose an epimorphism from a finitely generated free module  $F$  to  $M$  with kernel  $K$ . Since  $K$  is the union of its finitely generated submodules,  $M$  is the colimit of the directed system  $F/L$  where  $L$  runs over the finitely generated submodules of  $K$ . The functor  $\text{Tor}$  commutes in both variables with colimits over directed systems [10, Proposition VI.1.3. on page 107]. Now the claim follows from Step 1 and Theorem 2.9.

Step 3: If  $M$  is a  $\mathbb{C}\Gamma$ -module, then  $\dim_{\mathcal{N}(\Gamma)} (\text{Tor}_p^{\mathbb{C}\Gamma}(M, \mathcal{N}(\Gamma))) = 0$  for all  $p \geq 1$

We use induction over  $p \geq 1$ . The induction begin is already done in Step 2. Choose an exact sequence  $0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0$  of  $\mathcal{N}(\Gamma)$ -modules such that  $F$  is free. Then we obtain an isomorphism  $\text{Tor}_p^{\mathbb{C}\Gamma}(M, \mathcal{N}(\Gamma)) \cong \text{Tor}_{p-1}^{\mathbb{C}\Gamma}(N, \mathcal{N}(\Gamma))$  and the induction step follows. This finishes the proof of Theorem 5.1. ■

**Theorem 5.11** *Let  $\Gamma$  be an amenable group and  $X$  a  $\Gamma$ -space. Then*

$$b_p^{(2)}(X; \mathcal{N}(\Gamma)) = \dim_{\mathcal{N}(\Gamma)} (\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} H_p^{\text{sing}}(X; \mathbb{C}))$$



where  $H_p^{\text{sing}}(X; \mathbb{C})$  is the  $\mathbb{C}\Gamma$ -module given by the singular homology of  $X$  with complex coefficients. In particular  $b_p^{(2)}(X; \mathcal{N}(\Gamma))$  depends only on the  $\mathbb{C}\Gamma$ -module  $H_p^{\text{sing}}(X; \mathbb{C})$ .

**Proof** : We have to show for a  $\mathbb{C}\Gamma$ -chain complex  $C_*$  with  $C_p = 0$  for  $p < 0$

$$\dim(H_p(\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} C_*)) = \dim(\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} H_p(C_*)). \quad (5.12)$$

We begin with the case where  $C_*$  is projective. Then there is a universal coefficient spectral sequence converging to  $H_{p+q}(\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} C_*)$  [46, Theorem 5.6.4 on page 143] whose  $E^2$ -term is  $E_{p,q}^2 = \text{Tor}_p^{\mathbb{C}\Gamma}(H_q(C_*), \mathcal{N}(\Gamma))$ . Now Additivity of  $\dim_{\mathcal{N}(\Gamma)}$  (see Theorem 0.6.4) together with Theorem 5.1 imply 5.12 if  $C_*$  is projective.

Next we prove 5.12 in the case where  $C_*$  is acyclic. One reduces the claim to two-dimensional  $C_*$  and then checks this special case using long exact Tor-sequences, Additivity (see Theorem 0.6.4) and Theorem 5.1.

In the general case one chooses a projective  $\mathbb{C}\Gamma$ -chain complex  $P_*$  together with a  $\mathbb{C}\Gamma$ -chain map  $f_* : P_* \rightarrow C_*$  which induces an isomorphism on homology. Since the mapping cylinder is  $\mathbb{C}\Gamma$ -chain homotopy equivalent to  $C_*$ , the mapping cone of  $f_*$  is acyclic and hence 5.12 is true for  $P_*$  and the mapping cone, we get 5.12 for  $C_*$  from Additivity (see Theorem 0.6.4). This finishes the proof of Theorem 5.11. ■

We obtain as an immediate corollary from Theorem 4.10 and Theorem 5.11 (cf. [11, Theorem 0.2 on page 191]).

**Corollary 5.13** *If  $\Gamma$  is infinite amenable, then for all  $p \geq 0$*

$$b_p^{(2)}(\Gamma) = 0. \quad \blacksquare$$

**Remark 5.14** It is likely that Theorem 5.1 characterizes amenable groups. Namely, if  $\Gamma$  contains a free group  $F$  of rank two, then  $\Gamma$  is non-amenable and Theorem 5.1 becomes false because Theorem 3.3 implies

$$\begin{aligned} \dim_{\mathcal{N}(\Gamma)}(\text{Tor}_1^{\mathbb{C}\Gamma}(\mathbb{C}[\Gamma/F]; \mathcal{N}(\Gamma))) &= \dim_{\mathcal{N}(\Gamma)}(\text{Tor}_1^{\mathbb{C}F}(\mathbb{C}; \mathcal{N}(\Gamma))) \\ &= \dim_{\mathcal{N}(\Gamma)}((\mathcal{N}(\Gamma) \otimes_{\mathcal{N}(F)} \text{Tor}_1^{\mathbb{C}F}(\mathbb{C}; \mathcal{N}(F)))) \\ &= \dim_{\mathcal{N}(F)}(\text{Tor}_1^{\mathbb{C}F}(\mathbb{C}; \mathcal{N}(F))) \\ &= b_1^{(2)}(F) \\ &= -\chi(BF) \\ &= 1. \quad \blacksquare \end{aligned}$$

**Remark 5.15** In view of Theorem 5.1 the question arises when  $\mathcal{N}(\Gamma)$  is flat over  $\mathbb{C}\Gamma$ . Except for virtually cyclic groups, i.e. groups which are finite or contain an infinite cyclic subgroup of finite index, we know no examples of finitely presented groups  $\Gamma$  such that  $\mathcal{N}(\Gamma)$  is flat over  $\mathbb{C}\Gamma$ . If  $\mathcal{N}(\Gamma)$  is flat over  $\mathbb{C}\Gamma$ , then  $\mathcal{N}(\Delta)$  is flat over  $\mathbb{C}\Delta$  for any subgroup  $\Delta \subset \Gamma$  by

Theorem 3.3.1. Moreover,  $\mathcal{N}(\Gamma)$  is flat over  $\mathbb{C}\Gamma$  if and only if  $\mathcal{N}(\Delta)$  is flat over  $\mathbb{C}\Delta$  for any finitely generated subgroup  $\Delta \subset \Gamma$ . This follows from Theorem 3.3.1 and the facts that the functor  $\text{Tor}$  commutes in both variables with colimits over directed systems [10, Proposition VI.1.3. on page 107], any  $\mathbb{C}\Gamma$ -module is the colimit of its finitely generated submodules, any finitely generated  $\mathbb{C}\Gamma$ -module is the colimit of a directed system of finitely presented  $\mathbb{C}\Gamma$ -modules and any finitely presented  $\mathbb{C}\Gamma$ -submodule is obtained by induction from a finitely presented  $\mathbb{C}\Delta$ -module for a finitely generated subgroup  $\Delta \subset \Gamma$ . It is not hard to check that  $\mathcal{N}(\Gamma)$  is flat over  $\mathbb{C}\Gamma$ , if  $\mathcal{N}(\Delta)$  is flat over  $\mathbb{C}\Delta$  for some subgroup  $\Delta \subset \Gamma$  of finite index and that  $\mathcal{N}(\mathbb{Z})$  is flat over  $\mathbb{C}\mathbb{Z}$  using the fact that  $\mathbb{C}G$  is semi-simple for finite  $G$  and  $\mathbb{C}\mathbb{Z}$  is a principal ideal domain. In particular  $\mathcal{N}(\Gamma)$  is flat over  $\mathbb{C}\Gamma$  if  $\Gamma$  is virtually cyclic.

Now suppose that  $\mathcal{N}(\Gamma)$  is flat over  $\mathbb{C}\Gamma$ . If  $B\Gamma$  is a  $CW$ -complex of finite type, then  $b_p^{(2)}(E\Gamma; \mathcal{N}(\Gamma)) = 0$  for  $p \geq 1$  and the  $p$ -th Novikov-Shubin invariant satisfies  $\alpha_p(B\Gamma) = \infty^+$  for  $p \geq 2$ . This implies for instance that  $\Gamma$  does not contain a subgroup which is isomorphic to  $\mathbb{Z} * \mathbb{Z}$  (Remark 5.14) or  $\mathbb{Z} \times \mathbb{Z}$  [27, Proposition 39 on page 494]. If  $\Gamma$  is non-amenable and  $B\Gamma$  is a finite  $CW$ -complex, then  $B\Gamma$  is a counterexample to the zero-in-the-spectrum-conjecture [28], [35, section 11]. ■

## 6. Dimension functions and $G_0(\mathbb{C}\Gamma)$

Let  $G_0(\mathbb{C}\Gamma)$  be the abelian group which has as set of generators the isomorphism classes of finitely generated (not necessarily projective)  $\mathbb{C}\Gamma$ -modules and has for each exact sequence of finitely generated  $\mathbb{C}\Gamma$ -modules  $0 \longrightarrow M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow 0$  the relation  $[M_0] - [M_1] + [M_2] = 0$ . Given a finitely generated  $\mathbb{C}\Gamma$ -module  $M$ , the  $\mathcal{N}(\Gamma)$ -module  $\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M$  is a finitely generated  $\mathcal{N}(\Gamma)$ -module. We have defined  $\mathbf{T}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M$  and  $\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M$  in Definition 0.5. Recall from Theorem 0.6.3 that  $\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M$  is a finitely generated projective  $\mathcal{N}(\Gamma)$ -module. Define maps

$$i : K_0(\mathbb{C}\Gamma) \longrightarrow G_0(\mathbb{C}\Gamma) \quad [P] \mapsto [P]; \quad (6.1)$$

$$k : K_0(\mathbb{C}\Gamma) \longrightarrow K_0(\mathcal{N}(\Gamma)) \quad [P] \mapsto [\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} P]. \quad (6.2)$$

**Lemma 6.3** *If  $\Gamma$  is amenable, the map*

$$j : G_0(\mathbb{C}\Gamma) \longrightarrow K_0(\mathcal{N}(\Gamma)) \quad [M] \mapsto [\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M]$$

*is a well-defined homomorphism. The composition  $j \circ i$  agrees with  $k$  for the maps  $i$  and  $k$  defined in 6.1 and 6.2 above.*

**Proof** : If  $0 \longrightarrow M_0 \xrightarrow{i} M_1 \xrightarrow{p} M_2 \longrightarrow 0$  is an exact sequence of finitely generated  $\mathbb{C}\Gamma$ -modules we have to check in  $K_0(\mathcal{N}(\Gamma))$

$$[\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_0] - [\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_1] + [\mathbf{P}\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_2] = 0.$$

Consider the induced sequence  $\mathbf{PN}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_0 \xrightarrow{\bar{i}} \mathbf{PN}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_1 \xrightarrow{\bar{p}} \mathbf{PN}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_2$ . Obviously  $\bar{p}$  is surjective as  $p$  is surjective. We conclude from Theorem 0.6.1 that  $\ker(\bar{i})$  and  $\ker(\bar{p})$  are a finitely generated projective  $\mathcal{N}(\Gamma)$ -modules. Theorem 0.6.4 and Theorem 5.1 imply

$$\begin{aligned} \dim_{\mathcal{N}(\Gamma)}(\ker(\bar{i})) &= 0; \\ \dim_{\mathcal{N}(\Gamma)}(\text{im}(\bar{i})) &= \dim_{\mathcal{N}(\Gamma)}(\ker(\bar{p})). \end{aligned}$$

We conclude from Theorem 0.6 that  $\bar{i} : \mathbf{PN}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_0 \longrightarrow \ker(\bar{p})$  is a weak isomorphism, i.e. its kernel is trivial and  $\text{im}(\bar{i}) = \ker(\bar{p})$ . Since the functor  $\nu$  of Theorem 1.8 respects weak exactness and the Polar Decomposition Theorem applied to a weak isomorphism has an isomorphism as unitary part,  $\mathbf{PN}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_0$  and  $\ker(\bar{p})$  are isomorphic as  $\mathcal{N}(\Gamma)$ -modules. Since  $\ker(\bar{p}) \oplus \mathbf{PN}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_2$  and  $\mathbf{PN}(\Gamma) \otimes_{\mathbb{C}\Gamma} M_1$  are isomorphic, Lemma 6.3 follows.  $\blacksquare$

If we regard  $\text{hom}_{\mathbb{C}\Gamma}(M, \mathcal{N}(\Gamma))$  as left  $\mathcal{N}(\Gamma)$ -module by  $(af)(x) = f(x) \cdot a^*$  for  $a \in \mathcal{N}(\Gamma)$ ,  $f \in \text{hom}_{\mathbb{C}\Gamma}(M, \mathcal{N}(\Gamma))$  and  $x \in M$ , we obtain isomorphisms of  $\mathcal{N}(\Gamma)$ -modules

$$\begin{aligned} \text{hom}_{\mathbb{C}\Gamma}(M, \mathcal{N}(\Gamma)) &\xrightarrow{\cong} (\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M)^*; \\ (\mathbf{PN}(\Gamma) \otimes_{\mathbb{C}\Gamma} M)^* &\xrightarrow{\cong} (\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} M)^*. \end{aligned}$$

Since for a finitely generated projective  $\mathcal{N}(\Gamma)$ -module  $P$  its dual  $P^*$  is isomorphic to  $P$ , we conclude for a finitely generated  $\mathbb{C}\Gamma$ -module  $M$

$$j([M]) = [\text{hom}_{\mathbb{C}\Gamma}(M, \mathcal{N}(\Gamma))]. \quad (6.4)$$

For a finitely generated projective  $\mathcal{N}(\Gamma)$ -module  $P$  let

$$\dim_{\mathcal{N}(\Gamma)}^u(P) \in \text{cent}(\mathcal{N}(\Gamma)) \quad (6.5)$$

be its *center-valued von Neumann dimension* which is given in terms of the universal center-valued trace  $\text{tr}_{\mathcal{N}(\Gamma)}^u$  [24, Theorem 8.2.8 on page 517, Proposition 8.3.10 on page 525 and Theorem 8.4.3. on page 532], [34, section 3]. The center-valued von Neumann dimension is additive under direct sums and two finitely generated projective  $\mathcal{N}(\Gamma)$ -modules  $P$  and  $Q$  are isomorphic if and only if  $\dim_{\mathcal{N}(\Gamma)}^u(P) = \dim_{\mathcal{N}(\Gamma)}^u(Q)$ . We obtain an injection

$$\dim_{\mathcal{N}(\Gamma)}^u : K_0(\mathcal{N}(\Gamma)) \rightarrow \text{cent}(\mathcal{N}(\Gamma))^+ = \{a \in \text{cent}(\mathcal{N}(\Gamma)) \mid a = bb^* \text{ for } b \in \mathcal{N}(\Gamma)\}, \quad (6.6)$$

which is an isomorphism if  $\mathcal{N}(\Gamma)$  is of type II, for instance if  $\Gamma$  is finitely generated and does not contain an abelian subgroup of finite index ([34, Corollary 3.2 and Lemma 3.3]).

Next we investigate the relationship between  $K_0(\mathbb{C}\Gamma)$  and  $G_0(\mathbb{C}\Gamma)$  and between  $\dim_{\mathcal{N}(\Gamma)}^u$  and the Hattori-Stallings rank. Let  $\text{con}(\Gamma)$  be the set of conjugacy classes of elements in  $\Gamma$ . Let  $\text{con}(\Gamma)_f$  be the subset of  $\text{con}(\Gamma)$  of conjugacy classes  $(\gamma)$  for which each representative  $\gamma$  has finite order. Let  $\text{con}(\Gamma)_{cf}$  be the subset of  $\text{con}(\Gamma)$  of conjugacy classes  $(\gamma)$  which contain only finitely many elements. We denote by  $\text{class}_0(\Gamma)$  and  $\text{class}_0(\Gamma)_f$  respectively the complex vector space with the set  $\text{con}(\Gamma)$  and  $\text{con}(\Gamma)_f$  respectively as basis. We denote by  $\text{class}(\Gamma)$  and  $\text{class}(\Gamma)_{cf}$  respectively the complex vector space of functions from the set  $\text{con}(\Gamma)$  and

$\text{con}(\Gamma)_f$  respectively to  $\mathbb{C}$ . Notice that  $\text{class}_0(\Gamma)$  is the complex vector space of class functions from  $\Gamma$  to  $\mathbb{C}$  with finite support. Define the *universal  $\mathbb{C}\Gamma$ -trace* of  $\sum_{\gamma \in \Gamma} \lambda_\gamma \gamma \in \mathbb{C}\Gamma$  by

$$\text{tr}_{\mathbb{C}\Gamma}^u \left( \sum_{\gamma \in \Gamma} \lambda_\gamma \gamma \right) := \sum_{\gamma \in \Gamma} \lambda_\gamma \cdot (\gamma) \in \text{class}_0(\Gamma), \quad (6.7)$$

This extends to square matrices in the usual way

$$\text{tr}_{\mathbb{C}\Gamma}^u : M(n, n, \mathbb{C}\Gamma) \longrightarrow \text{class}_0(\Gamma) \quad A \mapsto \sum_{i=1}^n \text{tr}_{\mathbb{C}\Gamma}^u(a_{i,i}). \quad (6.8)$$

Let  $P$  be a finitely generated projective  $\mathbb{C}\Gamma$ -module. Define its Hattori-Stallings rank by

$$\text{HS}(P) := \text{tr}_{\mathbb{C}\Gamma}^u(A) \in \text{class}_0(\Gamma), \quad (6.9)$$

where  $A$  is any element in  $M(n, n, \mathbb{C}\Gamma)$  with  $A^2 = A$  such that the image of the map  $\mathbb{C}\Gamma^n \longrightarrow \mathbb{C}\Gamma^n$  given by right multiplication with  $A$  is  $\mathbb{C}\Gamma$ -isomorphic to  $P$ . This definition is independent of the choice of  $A$ . The Hattori-Stallings rank defines a homomorphism

$$\text{HS} : K_0(\mathbb{C}\Gamma) \longrightarrow \text{class}_0(\Gamma) \quad [P] \mapsto \text{HS}(P). \quad (6.10)$$

Define a homomorphism

$$\phi : \text{cent}(\mathcal{N}(\Gamma)) \longrightarrow \text{class}(\Gamma)_{cf} \quad (6.11)$$

by assigning to  $u \in \text{cent}(\mathcal{N}(\Gamma))$

$$\phi(u) : \text{con}(\Gamma)_{cf} \longrightarrow \mathbb{C} \quad (\delta) \mapsto \text{tr}_{\mathcal{N}(\Gamma)} \left( u \cdot \sum_{\delta' \in (\delta)} (\delta')^{-1} \right).$$

**Theorem 6.12** *Suppose that  $\Gamma$  is amenable. Then the following diagram commutes*

$$\begin{array}{ccccc} K_0(\mathbb{C}\Gamma) & & \xrightarrow{\text{HS}} & & \text{class}_0(\Gamma) \\ i \downarrow & & & & r \downarrow \\ G_0(\mathbb{C}\Gamma) & \xrightarrow{j} & K_0(\mathcal{N}(\Gamma)) & \xrightarrow{\dim_{\mathcal{N}(\Gamma)}^u} & \text{cent}(\mathcal{N}(\Gamma)) \xrightarrow{\phi} \text{class}(\Gamma)_{cf} \end{array}$$

where  $r$  is given by restriction and the other maps have been defined in 6.1, Lemma 6.3, 6.6, 6.10 and 6.11.

**Proof** : One has to show for an element  $A \in M(n, n, \mathbb{C}\Gamma)$  and  $\delta \in \Gamma$  such that  $(\delta)$  is finite

$$\text{tr}_{\mathbb{C}\Gamma}^u(A)(\delta) = (\phi \circ \text{tr}_{\mathcal{N}(\Gamma)}^u(A))(\delta). \quad (6.13)$$

It suffices to show for  $\gamma \in \Gamma$  and  $\delta \in \Gamma$  such that  $(\delta)$  is finite

$$\text{tr}_{\mathbb{C}\Gamma}^u(\gamma)(\delta) = (\phi \circ \text{tr}_{\mathcal{N}(\Gamma)}^u(\gamma))(\delta). \quad (6.14)$$

The universal center-valued von Neumann trace satisfies for  $\gamma \in \Gamma$

$$\mathrm{tr}_{\mathcal{N}(\Gamma)}^u(\gamma) = \begin{cases} |(\gamma)|^{-1} \cdot \sum_{\gamma' \in (\gamma)} \gamma' & \text{if } (\gamma) \text{ is finite} \\ 0 & \text{otherwise} \end{cases}.$$

This follows from the facts that  $\mathrm{tr}_{\mathcal{N}(\Gamma)}^u$  is ultraweakly continuous and the identity on the center of  $\mathcal{N}(\Gamma)$  and that  $\mathrm{tr}_{\mathcal{N}(\Gamma)}^u(\frac{1}{n} \cdot \sum_{i=1}^n \delta_i) = \mathrm{tr}_{\mathcal{N}(\Gamma)}^u(\delta)$  holds for elements  $\delta_1, \delta_2, \dots, \delta_n$  in  $(\delta)$ . Notice that  $\mathrm{tr}_{\mathcal{N}(\Gamma)}(\delta)$  is 1 if  $\delta = 1$  and 0 otherwise and that  $\mathrm{tr}_{\mathcal{C}\Gamma}^u(\gamma)(\delta) = 0$  if  $(\delta)$  is finite and  $(\gamma)$  is infinite. Hence 6.14 and thus 6.13 follow from the computation for  $\gamma, \delta \in \Gamma$  such that  $(\gamma)$  and  $(\delta)$  are finite

$$\begin{aligned} \phi(|(\gamma)|^{-1} \cdot \sum_{\gamma' \in (\gamma)} \gamma')(\delta) &= \mathrm{tr}_{\mathcal{N}(\Gamma)} \left( \left( |(\gamma)|^{-1} \cdot \sum_{\gamma' \in (\gamma)} \gamma' \right) \cdot \left( \sum_{\delta' \in (\delta)} (\delta')^{-1} \right) \right) \\ &= \sum_{\gamma' \in (\gamma)} \sum_{\delta' \in (\delta)} |(\gamma)|^{-1} \cdot \mathrm{tr}_{\mathcal{N}(\Gamma)}(\gamma' \cdot (\delta')^{-1}) \\ &= \sum_{\gamma' \in (\gamma), \delta' \in (\delta), \gamma' = \delta'} |(\gamma)|^{-1} \\ &= \begin{cases} 1 & \text{if } (\gamma) = (\delta) \\ 0 & \text{otherwise} \end{cases} \\ &= \mathrm{tr}_{\mathcal{C}\Gamma}^u(\gamma)(\delta). \end{aligned}$$

This finishes the proof of Theorem 6.12.  $\blacksquare$

**Lemma 6.15** *Let  $\Gamma$  be a discrete group. Then there is a commutative diagram whose the left vertical arrow is an isomorphism*

$$\begin{array}{ccc} (\mathrm{colim}_{\mathrm{Or}(\Gamma, \mathcal{FIN})} K_0(\mathbb{C}H)) \otimes_{\mathbb{Z}} \mathbb{C} & \xrightarrow{l} & K_0(\mathbb{C}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \\ h \downarrow \cong & & \downarrow \mathrm{HS} \\ \mathrm{class}_0(\Gamma)_f & \xrightarrow[e]{} & \mathrm{class}_0(\Gamma) \end{array}$$

**Proof :** Firstly we explain the maps in the square. The colimit is taken for the covariant functor

$$\mathrm{Or}(G, \mathcal{FIN}) \longrightarrow \mathrm{ABEL} \quad G/H \mapsto K_0(\mathbb{C}H)$$

to the category of abelian groups which is given by induction. Here  $\mathrm{Or}(G, \mathcal{FIN})$  is the full subcategory of the orbit category  $\mathrm{Or}(G)$  consisting of objects  $G/H$  with finite  $H$ . The map  $l$  is induced by the universal property of the colimit and the various maps  $K_0(\mathbb{C}H) \longrightarrow K_0(\mathbb{C}\Gamma)$  induced by the inclusions of finite subgroups  $H$  of  $\Gamma$  in  $\Gamma$ . The map  $e$  is given by the inclusion  $\mathrm{con}(\Gamma)_f \longrightarrow \mathrm{con}(\Gamma)$ .

Define for a group homomorphism  $\psi : \Gamma \longrightarrow \Gamma'$  a map  $\psi_* : \mathrm{con}(\Gamma) \longrightarrow \mathrm{con}(\Gamma')$  by sending  $(h)$  to  $\psi(h)$ . It induces a homomorphism  $\psi_* : \mathrm{class}_0(\Gamma) \longrightarrow \mathrm{class}_0(\Gamma')$ . One easily checks

that the following diagram commutes

$$\begin{array}{ccc}
K_0(\mathbb{C}\Gamma) & \xrightarrow{\psi_*} & K_0(\mathbb{C}\Gamma') \\
\text{HS} \downarrow & & \text{HS} \downarrow \\
\text{class}_0(\Gamma) & \xrightarrow{\psi_*} & \text{class}_0(\Gamma')
\end{array} \tag{6.16}$$

There is a canonical isomorphism

$$f_1 : (\text{colim}_{\text{Or}(\Gamma, \mathcal{FIN})} K_0(\mathbb{C}H)) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \text{colim}_{\text{Or}(\Gamma, \mathcal{FIN})} K_0(\mathbb{C}H) \otimes_{\mathbb{Z}} \mathbb{C}. \tag{6.17}$$

The Hattori-Stallings ranks for the various finite subgroups  $H$  of  $\Gamma$  induce an isomorphism

$$f_2 : \text{colim}_{\text{Or}(\Gamma, \mathcal{FIN})} K_0(\mathbb{C}H) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \text{colim}_{\text{Or}(\Gamma, \mathcal{FIN})} \text{class}(H). \tag{6.18}$$

Let  $f'_3 : \text{colim}_{\text{Or}(\Gamma, \mathcal{FIN})} \text{con}(H) \longrightarrow \text{con}(\Gamma)_f$  be the map induced by the inclusions of the finite subgroups  $H$  of  $\Gamma$ . Define a map  $f'_4 : \text{con}(\Gamma)_f \longrightarrow \text{colim}_{\text{Or}(\Gamma, \mathcal{FIN})} \text{con}(H)$  by sending  $(\gamma) \in \text{con}(\Gamma)_f$  to the image of  $(\gamma) \in \text{con}(\langle \gamma \rangle)$  under the canonical structure map from  $\text{con}(\langle \gamma \rangle)$  to  $\text{colim}_{\text{Or}(\Gamma, \mathcal{FIN})} \text{con}(H)$ , where  $\langle \gamma \rangle$  is the finite cyclic subgroup generated by  $\gamma$ . One easily checks that this is independent of the choice of the representative  $\gamma$  in  $(\gamma)$  and that  $f'_3$  and  $f'_4$  are inverse to one another. The bijection  $f'_3$  induces an isomorphism

$$f_3 : \text{colim}_{\text{Or}(\Gamma, \mathcal{FIN})} \text{class}(H) \longrightarrow \text{class}(\Gamma)_f, \tag{6.19}$$

because colimit and the functor sending a set to the complex vector space with this set as basis commute. Now the isomorphism  $h$  is defined as the composition of the isomorphisms  $f_1$  from 6.17,  $f_2$  from 6.18 and  $f_3$  from 6.19. It remains to check that the square in Lemma 6.15 commutes. This follows from the commutativity of 6.16. This finishes the proof of Lemma 6.15. ■

**Corollary 6.20** *Suppose that  $\Gamma$  is amenable. Then the image of the composition*

$$G_0(\mathbb{C}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{j \otimes_{\mathbb{Z}} \mathbb{C}} K_0(\mathcal{N}(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\dim_{\mathcal{N}(\Gamma)}^u} \text{cent}(\mathcal{N}(\Gamma)) \xrightarrow{\phi} \text{class}(\Gamma)_{cf}$$

*contains the complex vector space  $\text{class}_0(\Gamma)_{f,cf}$  with  $\text{con}(\Gamma)_f \cap \text{con}(\Gamma)_{cf}$  as basis.* ■

**Remark 6.21** There is the conjecture that the canonical map

$$\text{colim}_{\text{Or}(\Gamma, \mathcal{FIN})} K_0(\mathbb{C}H) \xrightarrow{\cong} K_0(\mathbb{C}\Gamma)$$

is bijective for all groups  $\Gamma$ . In particular this would imply by Lemma 6.15 that the Hattori-Stallings rank induces an isomorphism

$$\text{HS} : K_0(\mathbb{C}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \text{class}_0(\Gamma)_f. \quad \blacksquare$$

**Theorem 6.22** 1. *The map*

$$l : (\operatorname{colim}_{\operatorname{Or}(\Gamma, \mathcal{FIN})} K_0(\mathbb{C}H)) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow K_0(\mathbb{C}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C}$$

*is injective*

2. *If  $\Gamma$  is virtually polycyclic, then we obtain isomorphisms*

$$\begin{aligned} \operatorname{HS} : K_0(\mathbb{C}\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} &\xrightarrow{\cong} \operatorname{class}_0(\Gamma)_f; \\ i : K_0(\mathbb{C}\Gamma) &\xrightarrow{\cong} G_0(\mathbb{C}\Gamma). \end{aligned}$$

**Proof** : 1.) follows directly from Lemma 6.15.

2.) Moody has shown [38] that the obvious map  $\oplus_{H \in \mathcal{FIN}} G_0(\mathbb{C}H) \longrightarrow G_0(\mathbb{C}\Gamma)$  given by induction is surjective. Since  $\Gamma$  is polycyclic the complex group ring  $\mathbb{C}\Gamma$  is regular, i.e. noetherian and any  $\mathbb{C}\Gamma$ -module has a finite-dimensional projective resolution. Now 1.) and Lemma 6.15 prove the claim. ■

Theorem 6.22.2 has already been proven in [9].

**Remark 6.23** In particular we get from Theorem 6.12 that the map

$$\iota : \mathbb{Z} \longrightarrow G_0(\mathbb{C}\Gamma) \quad n \mapsto [\mathbb{C}\Gamma^n]$$

is split injective, provided that  $\Gamma$  is amenable. It is likely that this property characterizes amenable groups. At least we can show for a group  $\Gamma$  which contains the free group  $F_2$  in two letters as subgroup, that  $\iota$  is trivial by the following argument.

Induction with the inclusion  $F_2 \longrightarrow \Gamma$  induces a homomorphism  $G_0(\mathbb{C}F_2) \longrightarrow G_0(\mathbb{C}\Gamma)$  which sends  $[\mathbb{C}F_2]$  to  $[\mathbb{C}\Gamma]$ . Hence it suffices to show  $[\mathbb{C}F_2] = 0$  in  $G_0(\mathbb{C}F_2)$ . The cellular chain complex of the universal covering of  $S^1 \vee S^1$  yields an exact sequence of  $\mathbb{C}\Gamma$ -modules  $0 \rightarrow (\mathbb{C}F_2)^2 \rightarrow \mathbb{C}F_2 \rightarrow \mathbb{C} \rightarrow 0$ , where  $\mathbb{C}$  is equipped with the trivial  $F_2$ -action. Hence it suffices to show  $[\mathbb{C}] = 0$  in  $G_0(\mathbb{C}F_2)$ . Choose an epimorphism  $f : F_2 \longrightarrow \mathbb{Z}$ . Restriction with  $f$  defines a homomorphism  $G_0(\mathbb{C}\mathbb{Z}) \longrightarrow G_0(\mathbb{C}F_2)$ . It sends  $\mathbb{C}$  viewed as trivial  $\mathbb{C}\mathbb{Z}$ -module to  $\mathbb{C}$  viewed as trivial  $\mathbb{C}F_2$ -module. Hence it remains to show  $[\mathbb{C}] = 0$  in  $G_0(\mathbb{C}\mathbb{Z})$ . This follows from the exact sequence  $0 \longrightarrow \mathbb{C}\mathbb{Z} \xrightarrow{s-1} \mathbb{C}\mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow 0$  for  $s$  a generator of  $\mathbb{Z}$ . ■

## 7. Groups with vanishing $L^2$ -Betti numbers

In this section we investigate the following class of groups

**Definition 7.1** *Define the class of groups*

$$\begin{aligned} \mathcal{B}_d &:= \{ \Gamma \mid b_p^{(2)}(\Gamma) = 0 \text{ for } 0 \leq p \leq d \}; \\ \mathcal{B}_\infty &:= \{ \Gamma \mid b_p^{(2)}(\Gamma) = 0 \text{ for } 0 \leq p \}. \end{aligned} \quad \blacksquare$$

Notice that  $\mathcal{B}_0$  is the class of infinite groups by Theorem 4.10. Definition 7.1 is motivated among other things by Corollary 8.5 and the following result.

**Theorem 7.2** *Let  $1 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \pi$  be an exact sequence of groups. Suppose that  $\Gamma$  is finitely presented and one of the following conditions is satisfied.*

1.  $|\Delta| = \infty$ ,  $b_1^{(2)}(\Delta) < \infty$  and  $\pi$  contains an element of infinite order or contains finite subgroups of arbitrary large order;
2. The ordinary first Betti number of  $\Delta$  satisfies  $b_1(\Delta) < \infty$  and  $\pi$  belongs to  $\mathcal{B}_1$ .

*Then:*

1. Let  $M$  be a closed oriented 4-manifold with  $\Gamma$  as fundamental group. Then

$$|\text{sign}(M)| \leq \text{ch}(M);$$

2. Let  $\text{def}(\Gamma)$  be the deficiency, i.e. the maximum  $g(P) - r(P)$  for all presentations  $P$  where  $g(P)$  is the number of generators and  $r(P)$  the number of relations. Then

$$\text{def}(\Gamma) \leq 1;$$

**Proof** : If the first condition is satisfied, then  $\Gamma$  belongs to  $\mathcal{B}_1$  by Theorem 7.3.5. Now apply [32, Theorem 5.1 and Theorem 6.1 on page 212] .

Suppose that the second condition is satisfied. Let  $p : \overline{M} \longrightarrow M$  be the regular covering associated to  $\Delta$ . There is a universal coefficient spectral sequence converging to  $H_{p+q}^\pi(\overline{M}; \mathcal{N}(\pi))$  with  $E_{p,q}^2 = \text{Tor}_p^{\mathbb{C}\pi}(H_q(\overline{M}; \mathbb{C}), \mathcal{N}(\pi))$  [46, Theorem 5.6.4 on page 143]. Since  $H_q(\overline{M}; \mathbb{C})$  is  $\mathbb{C}$  with the trivial  $\pi$ -action for  $q = 0$  and finite-dimensional as complex vector space by assumption for  $q = 1$ , Theorem 0.6.4 and Lemma 3.4.3 imply  $\dim(E_{p,q}^2) = 0$  for  $p + q = 1$  and hence  $b_1^{(2)}(\overline{M}; \mathcal{N}(\pi)) = 0$ . The arguments in [32, Theorem 5.1 and Theorem 6.1 on page 212] for the universal covering of  $M$  apply also to  $\overline{M}$ . ■

The idea to take another covering than the universal covering in the proof of Theorem 7.2 is taken from [17, Corollary 5.2 on page 391]. More information about results like Theorem 7.2 are given can be found in [18].

**Theorem 7.3** *Let  $d$  be a non-negative integer or  $d = \infty$ . Then:*

1. The class  $\mathcal{B}_\infty$  contains all infinite amenable groups;
2. If  $\Gamma$  contains a normal subgroup  $\Delta$  with  $\Delta \in \mathcal{B}_d$ , then  $\Gamma \in \mathcal{B}_d$ ;
3. If  $\Gamma$  is the union of a directed system of subgroups  $\{\Gamma_i \mid i \in I\}$  such that each  $\Gamma_i$  belongs to  $\mathcal{B}_d$ , then  $\Gamma \in \mathcal{B}_d$ ;



4. Let  $1 \longrightarrow \Delta \xrightarrow{i} \Gamma \xrightarrow{p} \pi \longrightarrow 1$  be an exact sequence of groups such that  $b_p^{(2)}(\Delta)$  is finite for all  $p \leq d$ . Suppose that  $B\pi$  has finite  $d$ -skeleton and that there is an injective endomorphism  $j : \pi \longrightarrow \pi$  whose image has finite index, but is not equal to  $\pi$  (for example  $\pi = \mathbb{Z}^n$ ). Then  $\Gamma \in \mathcal{B}_d$ ;
5. Let  $1 \longrightarrow \Delta \xrightarrow{i} \Gamma \xrightarrow{p} \pi \longrightarrow 1$  be an exact sequence of groups such that  $\Delta \in \mathcal{B}_{d-1}$ ,  $b_d^{(2)}(\Delta) < \infty$  and  $\Gamma$  contains an element of infinite order or a finite subgroup of arbitrary large order. Then  $\Gamma \in \mathcal{B}_d$ ;
6. Suppose that there are groups  $\Gamma_1$  and  $\Gamma_2$  and group homomorphisms  $\phi_i : \Gamma_0 \longrightarrow \Gamma_i$  for  $i = 1, 2$  such that  $\phi_0$  and  $\phi_1$  are injective,  $\Gamma_0$  belongs to  $\mathcal{B}_{d-1}$ ,  $\Gamma_1$  and  $\Gamma_2$  belong to  $\mathcal{B}_d$  and  $\Gamma$  is the amalgamated product  $\Gamma_1 *_{\Gamma_0} \Gamma_2$  with respect to  $\phi_1$  and  $\phi_2$ . Then  $\Gamma$  belongs to  $\mathcal{B}_d$ .

**Proof :** 1.) see Corollary 5.13.

2.) We obtain a fibration  $B\Delta \longrightarrow B\Gamma \longrightarrow B\pi$  for  $\pi = \Gamma/\Delta$ . There is the Leray-Serre spectral sequence converging to  $H_{p+q}^\Gamma(E\Gamma, \mathcal{N}(\Gamma))$  with  $E_{p,q}^1 = H_q^\Delta(E\Delta; \mathcal{N}(\Gamma)) \otimes_{\mathbb{Z}\pi} C_p(E\pi)$  for an appropriate  $\mathbb{Z}\pi$ -action on  $H_q^\Delta(E\Delta; \mathcal{N}(\Gamma))$  coming from the fiber transport. Because of Additivity (see Theorem 0.6.4) it suffices to show for  $p + q \leq d$

$$\dim_{\mathcal{N}(\Gamma)} (E_{p,q}^1) = 0. \quad (7.4)$$

Since  $b_q^{(2)}(\Delta) = \dim_{\mathcal{N}(\Gamma)} (H_q^\Delta(E\Delta; \mathcal{N}(\Gamma)))$  by Theorem 3.3 and  $C_p(E\pi)$  is a direct sum of copies of  $\mathbb{Z}\pi$ , Cofinality (see Theorem 0.6.4) proves 7.4.

3.) Using for instance the bar-resolution model for  $E\Gamma$ , one gets that  $E\Gamma$  is the colimit of a directed system of subspaces of the form  $E\Gamma_i \times_{\Gamma_i} \Gamma$  directed by  $I$ . Hence

$$H_p^\Gamma(E\Gamma; \mathcal{N}(\Gamma)) = \operatorname{colim}_{i \in I} H_p^\Gamma(E\Gamma_i \times_{\Gamma_i} \Gamma; \mathcal{N}(\Gamma)).$$

Since  $\dim_{\mathcal{N}(\Gamma)} (H_p^\Gamma(E\Gamma_i \times_{\Gamma_i} \Gamma; \mathcal{N}(\Gamma))) = b_p^{(2)}(\Gamma_i)$  by Theorem 4.9 the claim follows from Theorem 2.9.

4.) Fix an integer  $n \geq 1$ . Put  $\Gamma' = p^{-1}(\operatorname{im}(j^n))$ . If  $k$  is the index of  $\operatorname{im}(j)$  in  $\pi$ , then  $k^n$  is the index of  $\operatorname{im}(j^n)$  in  $\pi$  and of  $\Gamma'$  in  $\Gamma$  and we conclude

$$b_p^{(2)}(\Gamma) = \frac{b_p^{(2)}(\Gamma')}{k^n}. \quad (7.5)$$

Since  $\operatorname{im}(j^n)$  is isomorphic to  $\pi$ , we have an exact sequence  $1 \longrightarrow \Delta \longrightarrow \Gamma' \longrightarrow \pi \longrightarrow 1$ . Let  $i_p$  be the number of  $p$ -cells in  $B\pi$ . We get from the Leray-Serre spectral sequence and Additivity (see Theorem 0.6)

$$b_p^{(2)}(\Gamma') \leq \sum_{n=0}^p b_q^{(2)}(\Delta) \cdot i_{p-q}. \quad (7.6)$$

Equations 7.5 and 7.6 imply

$$b_p^{(2)}(\Gamma) = \frac{\sum_{q=0}^p b_q^{(2)}(\Delta) \cdot i_{p-q}}{k^n}. \quad (7.7)$$

Since  $k > 1$  and 7.7 holds for all  $n \geq 1$  and  $\sum_{q=0}^p b_q^{(2)}(\Delta) \cdot i_{p-q}$  is finite for  $p \leq d$  by assumption, the claim follows.

5.) Using the spectral sequence which converges to  $H_{p+q}^\Gamma(E\Gamma; \mathcal{N}(\Gamma))$  and has as  $E^2$ -term  $E_{p,q}^2 = H_p^\pi(E\pi; H_q^\Delta(E\Delta; \mathcal{N}(\Gamma)))$  the proof of assertion 5.) is reduced to the proof of

$$\dim_{\mathcal{N}(\Gamma)} (H_0^\pi(E\pi; H_d^\Delta(E\Delta; \mathcal{N}(\Gamma)))) = 0, \quad (7.8)$$

since  $\dim_{\mathcal{N}(\Gamma)} (H_q^\Delta(E\Delta; \mathcal{N}(\Gamma))) = b_q^{(2)}(\Delta)$  by Theorem 3.3 and hence vanishes for  $q < d$  by assumption. Let  $\pi' \subset \pi$  be a subgroup (not necessarily normal). Let  $\Gamma' \subset \Gamma$  be the preimage of  $\pi$  under the canonical projection  $\Gamma \rightarrow \pi$ . Then we obtain an exact sequence  $1 \rightarrow \Delta \rightarrow \Gamma' \rightarrow \pi' \rightarrow 1$ . We have

$$\begin{aligned} H_0^{\pi'}(E\pi'; H_d^\Delta(E\Delta; \mathcal{N}(\Gamma'))) &= H_d^\Delta(E\Delta; \mathcal{N}(\Gamma')) \otimes_{\mathbb{C}[\pi']} \mathbb{C}; \\ H_0^\pi(E\pi; H_d^\Delta(E\Delta; \mathcal{N}(\Gamma))) &= H_d^\Delta(E\Delta; \mathcal{N}(\Gamma)) \otimes_{\mathbb{C}[\pi]} \mathbb{C}; \end{aligned}$$

Since  $H_d^\Delta(E\Delta; \mathcal{N}(\Gamma)) \otimes_{\mathbb{C}[\pi]} \mathbb{C}$  is a quotient of  $H_d^\Delta(E\Delta; \mathcal{N}(\Gamma)) \otimes_{\mathbb{C}[\pi']} \mathbb{C}$  we conclude from Additivity (see Theorem 0.6) and from Theorem 3.3

$$\begin{aligned} \dim_{\mathcal{N}(\Gamma)} (H_d^\Delta(E\Delta; \mathcal{N}(\Gamma)) \otimes_{\mathbb{C}[\pi]} \mathbb{C}) &\leq \dim_{\mathcal{N}(\Gamma)} (H_d^\Delta(E\Delta; \mathcal{N}(\Gamma)) \otimes_{\mathbb{C}[\pi']} \mathbb{C}); \\ \dim_{\mathcal{N}(\Gamma')} (H_d^\Delta(E\Delta; \mathcal{N}(\Gamma')) \otimes_{\mathbb{C}[\pi']} \mathbb{C}) &= \dim_{\mathcal{N}(\Gamma)} (H_d^\Delta(E\Delta; \mathcal{N}(\Gamma)) \otimes_{\mathbb{C}[\pi']} \mathbb{C}). \end{aligned}$$

This implies

$$\dim_{\mathcal{N}(\Gamma)} (H_0^\pi(E\pi; H_d^\Delta(E\Delta; \mathcal{N}(\Gamma)))) \leq \dim_{\mathcal{N}(\Gamma')} (H_0^{\pi'}(E\pi'; H_d^\Delta(E\Delta; \mathcal{N}(\Gamma')))).$$

Hence 7.8 would follow if we can find for each  $\epsilon < 0$  a subgroup  $\pi' \subset \pi$  satisfying

$$\dim_{\mathcal{N}(\Gamma')} (H_0^{\pi'}(E\pi'; H_d^\Delta(E\Delta; \mathcal{N}(\Gamma')))) \leq \epsilon. \quad (7.9)$$

We begin with the case where  $\pi'$  is  $\mathbb{Z}$ . From assertion 4 we conclude

$$\dim_{\mathcal{N}(\Gamma')} (H_p^{\Gamma'}(E\Gamma'; \mathcal{N}(\Gamma'))) = 0 \quad \text{for } p \geq n. \quad (7.10)$$

The Leray-Serre spectral sequence associated to  $1 \rightarrow \Delta \rightarrow \Gamma' \rightarrow \mathbb{Z} \rightarrow 1$  has an  $E^2$ -term which satisfies  $E_{p,q}^2 = 0$  for  $q \neq 0, 1$  since  $B\mathbb{Z}$  has the 1-dimensional model  $S^1$ . Since it converges to  $H_{p+q}^{\Gamma'}(E\Gamma'; \mathcal{N}(\Gamma'))$ , we conclude 7.9 for  $\epsilon = 0$  from 7.10 and Additivity (see Theorem 0.6). Now suppose  $\pi'$  is finite. Then we get

$$\begin{aligned} \dim_{\mathcal{N}(\Gamma')} (H_0^{\pi'}(E\pi'; H_d^\Delta(E\Delta; \mathcal{N}(\Gamma')))) &= \dim_{\mathcal{N}(\Gamma')} (H_d^{\Gamma'}(E\Gamma'; \mathcal{N}(\Gamma'))) \\ &= b_d^{(2)}(\Gamma') \\ &= \frac{b_d^{(2)}(\Delta)}{|\pi'|}. \end{aligned}$$

If we can find  $\pi'$  with arbitrary large  $|\pi'|$  we get 7.9.

6.) One easily checks using the Seifert-van Kampen Theorem, that there is a  $\Gamma$ -push out

$$\begin{array}{ccc}
\Gamma \times_{\Gamma_0} E\Gamma_0 & \longrightarrow & \Gamma \times_{\Gamma_1} E\Gamma_1 \\
\downarrow & & \downarrow \\
\Gamma \times_{\Gamma_2} E\Gamma_2 & \longrightarrow & E\Gamma
\end{array}$$

We conclude from Theorem 4.9  $\dim(H_p^\Gamma(E\Gamma_i \times_{\Gamma_i} \Gamma; \mathcal{N}(\Gamma))) = b_p^{(2)}(\Gamma_i)$ . Now the claim follows from Additivity (see Theorem 0.6.4) and the a long exact homology sequence for  $H_*^\Gamma(-, \mathcal{N}(\Gamma))$ . This finishes the proof of Theorem 7.3. ■

So far we have no example with negative answer to the following question and can give an affirmative answer in some special cases.

**Question 7.11** Let  $1 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \pi \longrightarrow 1$  be an exact sequence such that  $b_p^{(2)}(\Delta) < \infty$  for all  $p \geq 0$  and  $\pi$  belongs to  $\mathcal{B}_\infty$ . Does then  $\Gamma$  belong to  $\mathcal{B}_\infty$ ?

More generally, if  $F \longrightarrow E \longrightarrow B$  is a fibration such that  $b_p^{(2)}(F; \mathcal{N}(\pi_1(E))) < \infty$  and  $b_p^{(2)}(B; \mathcal{N}(\pi_1(B))) = 0$  holds for  $p \geq 0$ , does then  $b_p^{(2)}(E; \mathcal{N}(\pi_1(E))) = 0$  hold for  $p \geq 0$ ? ■

**Remark 7.12** Compact 3-manifolds whose fundamental groups belong to  $\mathcal{B}_\infty$  are characterized in [29, Proposition 6.5 on page 54]. The generalized Singer-Conjecture says that for an aspherical closed manifold  $M$  all the  $L^2$ -Betti numbers of its universal covering vanish possibly except in the middle dimension. In particular it implies that the fundamental group of an aspherical closed odd-dimensional manifold belongs to  $\mathcal{B}_\infty$ . Thompson's group  $F$  belongs to  $\mathcal{B}_\infty$  [34, Theorem 0.8]. More information about the class  $\mathcal{B}_1$  is given in [4]. ■

## 8. $L^2$ -Euler characteristics and the Burnside group

In this section we extend some of the results [11] about  $L^2$ -Euler characteristics and investigate the Burnside group of a discrete group  $\Gamma$ . This extends the classical notions of the Burnside ring, Burnside ring congruences and equivariant Euler characteristics for finite groups.

If  $X$  is a  $\Gamma$ -CW-complex, denote by  $I(X)$  the set of its equivariant cells. For a cell  $c \in I(X)$  let  $(\Gamma_c)$  be the conjugacy class of subgroups of  $\Gamma$  given by its orbit type and let  $\dim(c)$  be its dimension. Denote by  $|\Gamma_c|^{-1}$  the inverse of the order of any representative of  $(\Gamma_c)$ , where  $|\Gamma_c|^{-1}$  is to be understood to be zero if the order is infinite.

**Definition 8.1** Let  $X$  be a (left)  $\Gamma$ -space and  $V$  be a  $\mathcal{A}$ - $\mathbb{Z}\Gamma$ -module. Define

$$\begin{aligned}
h(X; V) &:= \sum_{p \geq 0} b_p^{(2)}(X; V) \in [0, \infty]; \\
\chi^{(2)}(X; V) &:= \sum_{p \geq 0} (-1)^p \cdot b_p^{(2)}(X; V) \in \mathbb{R} \quad , \text{if } h(X; V) < \infty; \\
m(X) &:= \sum_{c \in I(X)} |\Gamma_c|^{-1} \in [0, \infty], \quad , \text{if } X \text{ is a } \Gamma\text{-CW-complex.} \quad \blacksquare
\end{aligned}$$

The condition  $h(X; V) < \infty$  ensures that the sum defining  $\chi^{(2)}(X; V)$  converges and that  $\chi^{(2)}(X; V)$  satisfies the usual additivity formula, i.e. for a  $\Gamma$ -CW-complex  $X$  with  $\Gamma$ -CW-subcomplexes  $X_0$ ,  $X_1$  and  $X_2$  satisfying  $X = X_1 \cup X_2$ ,  $X_0 = X_1 \cap X_2$  and  $h(X_k; V) < \infty$  for  $k = 0, 1, 2$  one has

$$h(X; V) < \infty; \quad (8.2)$$

$$\chi^{(2)}(X; V) = \chi^{(2)}(X_1; V) + \chi^{(2)}(X_2; V) - \chi^{(2)}(X_0; V). \quad (8.3)$$

The next theorem generalizes [11, Theorem 0.3 on page 191]

**Theorem 8.4** *Let  $X$  and  $Y$  be  $\Gamma$ -CW-complexes such that  $m(X) < \infty$  and  $m(Y) < \infty$  holds. Then*

1.

$$\begin{aligned} h(X; \mathcal{N}(\Gamma)) &< \infty; \\ \sum_{c \in I(X)} (-1)^{\dim(c)} \cdot |\Gamma_c|^{-1} &= \chi^{(2)}(X; \mathcal{N}(\Gamma)). \end{aligned}$$

2. Suppose that  $\Gamma$  is amenable. Then

$$\sum_{c \in I(X)} (-1)^{\dim(c)} \cdot |\Gamma_c|^{-1} = \sum_{p \geq 0} (-1)^p \cdot \dim(\mathcal{N}(\Gamma) \otimes_{\mathbb{C}\Gamma} H_p(X; \mathbb{C})),$$

where  $H_p(X; \mathbb{C})$  is the cellular or the singular homology of  $X$  with complex coefficients. In particular  $\sum_{c \in I(X)} (-1)^{\dim(c)} \cdot |\Gamma_c|^{-1}$  depends only the  $\mathbb{C}\Gamma$ -isomorphism class of the  $\mathbb{C}\Gamma$ -modules  $H_n(X; \mathbb{C})$  for all  $n \geq 0$ ;

3. If for all  $c \in I(X)$  the group  $\Gamma_c$  is finite or belongs to the class  $\mathcal{B}_\infty$ , then

$$\begin{aligned} b_p^{(2)}(X; \mathcal{N}(\Gamma)) &= b_p^{(2)}(E\Gamma \times X; \mathcal{N}(\Gamma)) \quad \text{for } p \geq 0; \\ \chi^{(2)}(X; \mathcal{N}(\Gamma)) &= \chi^{(2)}(E\Gamma \times X; \mathcal{N}(\Gamma)); \\ \sum_{c \in I(X)} (-1)^{\dim(c)} \cdot |\Gamma_c|^{-1} &= \chi^{(2)}(E\Gamma \times X; \mathcal{N}(\Gamma)). \end{aligned}$$

4. Suppose that  $f : X \longrightarrow Y$  is a  $\Gamma$ -equivariant map, such that the induced map  $H_p(f; \mathbb{C})$  on the singular or cellular homology with complex coefficients is bijective. Suppose that for all  $c \in I(X)$  and  $c \in I(Y)$  the group  $\Gamma_c$  is finite or belongs to the class  $\mathcal{B}_\infty$ . Then

$$\sum_{c \in I(X)} (-1)^{\dim(c)} \cdot |\Gamma_c|^{-1} = \sum_{c \in I(Y)} (-1)^{\dim(c)} \cdot |\Gamma_c|^{-1}.$$

**Proof :** 1.) Additivity and Cofinality (see Theorem 0.6) and Lemma 3.4.1 imply

$$\begin{aligned} \dim(C_p(X; \mathcal{N}(\Gamma))) &= \sum_{c \in I(X), \dim(c)=p} |\Gamma_c|^{-1}; \\ \dim(H_p^\Gamma(X; \mathcal{N}(\Gamma))) &\leq \dim(C_p(X; \mathcal{N}(\Gamma))); \\ \sum_{p \geq 0} (-1)^p \cdot \dim(C_p(X; \mathcal{N}(\Gamma))) &= \chi^{(2)}(X; \mathcal{N}(\Gamma)). \end{aligned}$$

2.) follows from the first assertion and Theorem 5.11.

3.) Because of the first equation it suffices to prove that the dimension of the kernel and the cokernel of the map induced by the projection

$$\mathrm{pr}_* : H_p^\Gamma(E\Gamma \times X; \mathcal{N}(\Gamma)) \longrightarrow H_p^\Gamma(X; \mathcal{N}(\Gamma))$$

are trivial. Notice that  $X$  is the colimit of its finite  $\Gamma$ -subcomplexes. Since  $H_p^\Gamma(-, \mathcal{N}(\Gamma))$  is compatible with colimits and colimit preserves exact sequences, we can assume by Theorem 2.9 and Additivity (see Theorem 0.6) that  $X$  itself is finite. By induction over the number of equivariant cells, the long exact homology sequence and Additivity (see Theorem 0.6) the claim reduces to the case where  $X$  is of the shape  $\Gamma/H$ . Because of Theorem 4.9 it suffices to prove for the map  $\mathrm{pr}_* : H_p^H(EH; \mathcal{N}(H)) \longrightarrow H_p^H(\{*\}; \mathcal{N}(H))$  that its kernel and cokernel have trivial dimension, provided that  $H$  is finite or belongs to  $\mathcal{B}_\infty$ . This is obvious for finite  $H$  and follows for  $H \in \mathcal{B}_\infty$  from the definition of  $\mathcal{B}_\infty$  and Theorem 4.10.

4.) Since  $E\Gamma \times X$  is free and the map  $\mathrm{id} \times f : E\Gamma \times X \longrightarrow E\Gamma \times Y$  induces an isomorphism on singular homology it induces an isomorphism  $H_p^\Gamma(E\Gamma \times X; \mathcal{N}(\Gamma)) \longrightarrow H_p^\Gamma(E\Gamma \times Y; \mathcal{N}(\Gamma))$  and we conclude  $\chi^{(2)}(E\Gamma \times X; \mathcal{N}(\Gamma)) = \chi^{(2)}(E\Gamma \times Y; \mathcal{N}(\Gamma))$ . Now assertion 4.) follows from assertion 3.). This finishes the proof of Theorem 8.4. ■

As explained in [11, Proposition 0.4 on page 192] the  $L^2$ -Euler characteristic extends the notion of the virtual Euler characteristic which is due to Wall [44]. Information about this notion can be found for instance in [8, chapter IX]. The next result generalizes [11, Corollary 0.6 on page 193]

**Corollary 8.5** *Let  $\Gamma$  be a group belonging to  $\mathcal{B}_\infty$ . Then  $\chi^2(E\Gamma; \mathcal{N}(\Gamma))$  is defined and vanishes. If its virtual Euler characteristic  $\chi_{\mathrm{virt}}(\Gamma)$  is defined, then it vanishes. In particular  $\chi(B\Gamma)$  vanishes if  $B\Gamma$  can be chosen to be a finite CW-complex.* ■

Next we introduce the Burnside group and the equivariant Euler characteristic. The elementary proof the following lemma is left to the reader.

**Lemma 8.6** *Let  $H$  and  $K$  be subgroups of  $\Gamma$ . Then*

$$1. \Gamma/H^K = \{gH \mid g^{-1}Kg \subset H\};$$

2. The map

$$\phi : \Gamma/H^K \longrightarrow \mathrm{consub}(H) \quad gH \mapsto g^{-1}Kg$$

induces an injection

$$WK \backslash (\Gamma/H^K) \longrightarrow \mathrm{consub}(H),$$

where  $\mathrm{consub}(H)$  is the set of conjugacy classes in  $H$  of subgroups of  $H$ ;

3. The  $WK$ -isotropy group of  $gH \in \Gamma/H^K$  is  $(gHg^{-1} \cap NK)/K \subset NK/K = WK$ , where  $NK$  is the normalizer of  $K$  in  $\Gamma$  and  $WK$  is  $NK/K$ ;

4. If  $H$  is finite, then  $\Gamma/H^K$  is a finite union of  $WK$ -orbits of the shape  $WK/L$  for finite subgroups  $L \subset WK$ . ■

**Definition 8.7** Define the Burnside group  $A(\Gamma)$  by the Grothendieck group of the abelian monoid under disjoint union of  $\Gamma$ -isomorphism classes of proper cocompact  $\Gamma$ -sets  $S$ , i.e.  $\Gamma$ -sets  $S$  for which the isotropy group of each element in  $S$  and the quotient  $\Gamma \backslash S$  are finite. ■

Notice that  $A(\Gamma)$  is the free abelian group generated by  $\Gamma$ -isomorphism classes of orbits  $\Gamma/H$  for finite subgroups  $H \subset \Gamma$  and that  $\Gamma/H$  and  $\Gamma/K$  are  $\Gamma$ -isomorphic if and only if  $H$  and  $K$  are conjugate in  $\Gamma$ . If  $\Gamma$  is a finite group,  $A(\Gamma)$  is the classical Burnside ring [13, section 5], [14, chapter IV]. If  $\Gamma$  is infinite, then the cartesian product of two proper cocompact  $\Gamma$ -sets with the diagonal action is not cocompact any more so that the cartesian product does not induce a ring structure on  $A(\Gamma)$ . At least there is a bilinear map induced by the cartesian product  $A(\Gamma_1) \otimes A(\Gamma_2) \longrightarrow A(\Gamma_1 \times \Gamma_2)$ .

**Definition 8.8** Let  $X$  be a proper finite  $\Gamma$ -CW-complex. Define its equivariant Euler characteristic

$$\chi^\Gamma(X) := \sum_{c \in I(X)} (-1)^{\dim(c)} \cdot [\Gamma/\Gamma_c] \in A(\Gamma). \quad \blacksquare$$

An *additive invariant*  $(A, a)$  for proper finite  $\Gamma$ -CW-complexes  $X$  consists of an abelian group  $A$  and a function  $a$  which assigns to any proper finite  $\Gamma$ -CW-complex  $X$  an element  $a(X) \in A$  such that the following three conditions hold, i.) if  $X$  and  $Y$  are  $\Gamma$ -homotopy equivalent, then  $a(X) = a(Y)$ , ii.) if  $X_0, X_1$  and  $X_2$  are  $\Gamma$ -CW-subcomplexes of  $X$  with  $X = X_1 \cup X_2$  and  $X_0 = X_1 \cap X_2$ , then  $a(X) = a(X_1) + a(X_2) - a(X_0)$ , and iii.)  $a(\emptyset) = 0$ . We call an additive invariant  $(U, u)$  *universal*, if for any additive invariant  $(A, a)$  there is precisely one homomorphism  $\psi : U \longrightarrow A$  such that  $\psi(u(X)) = a(X)$  holds for all proper finite  $\Gamma$ -CW-complexes. One easily checks using induction over the number of equivariant cells

**Lemma 8.9**  $(A(\Gamma), \chi^\Gamma)$  is the universal additive invariant for finite proper  $\Gamma$ -CW-complexes. ■

**Definition 8.10** Define for a finite subgroup  $K \subset \Gamma$  the  $L^2$ -character map

$$\text{ch}_K^\Gamma : A(\Gamma) \longrightarrow \mathbb{Q} \quad [S] \mapsto \sum_{i=1}^r |L_i|^{-1}$$

if  $WK/L_1, WK/L_2, \dots, WK/L_r$  are the  $WK$ -orbits of  $S^K$ . Define the global  $L^2$ -character map by

$$\text{ch}^\Gamma := \prod_{(K)} \text{ch}_K^\Gamma : A(\Gamma) \longrightarrow \prod_{(K)} \mathbb{Q}$$

where  $(K)$  runs over the conjugacy classes of finite subgroups of  $\Gamma$ . ■

**Lemma 8.11** *Let  $X$  be a finite proper  $\Gamma$ -CW-complex and  $K \subset \Gamma$  be a finite subgroup. Then  $X^K$  is a finite proper WK-CW-complex and*

$$\chi^{(2)}(X^K; \mathcal{N}(WK)) = \text{ch}_K^\Gamma(\chi^\Gamma(X)).$$

**Proof :** The WK-space  $X^K$  is a finite proper WK-CW-complex because for finite  $H \subset \Gamma$  the WK-set  $\Gamma/H^K$  is proper and cocompact by Lemma 8.6. Since the assignment which associates to a finite proper  $\Gamma$ -CW-complex  $X$  the element  $\chi^{(2)}(X^K; \mathcal{N}(WK))$  in  $\mathbb{Q}$  is an additive invariant, it suffices by Lemma 8.9 to check the claim for  $X = \Gamma/H$  for finite  $H \subset \Gamma$ . Then the claim follows from the fact that  $\chi^{(2)}(WK/L; \mathcal{N}(WK)) = |L|^{-1}$  holds for finite  $L \subset WK$ . ■

Notice that one gets from Lemma 8.6 the following explicit formula for the value of  $\text{ch}_K^\Gamma(\Gamma/H)$ . Namely, define

$$\mathcal{L}_K(H) := \{(L) \in \text{consub}(H) \mid L \text{ conjugated to } K \text{ in } \Gamma\}.$$

For  $(L) \in \mathcal{L}_K(H)$  choose  $L \in (L)$  and  $g \in \Gamma$  with  $g^{-1}Kg = L$ . Then

$$\begin{aligned} g(H \cap NL)g^{-1} &= gHg^{-1} \cap NK; \\ |(gHg^{-1} \cap NK)/K|^{-1} &= \frac{|K|}{|H \cap NL|}. \end{aligned}$$

This implies

$$\text{ch}_K^\Gamma(\Gamma/H) = \sum_{L \in \mathcal{L}_K(H)} \frac{|K|}{|H \cap NL|}. \quad (8.12)$$

**Lemma 8.13** *The global  $L^2$ -character map of Definition 8.10 induces a map denoted by*

$$\text{ch}^\Gamma \otimes_{\mathbb{Z}} \mathbb{Q} : A(\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \prod_{(K)} \mathbb{Q}.$$

*It is injective. If  $\Gamma$  has only finitely many conjugacy classes of finite subgroups, then it is bijective.*

**Proof :** Consider an element  $\sum_{i=1}^n r_i \cdot [\Gamma/H_i]$  in the kernel of  $\text{ch}^\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ . We show by induction over  $n$  that the element must be trivial. The begin  $n = 0$  is trivial, the induction step done as follows. We can choose the numeration such that  $H_i$  subconjugated to  $H_j$  implies  $i \geq j$ . We get from 8.12

$$\begin{aligned} \text{ch}_K^\Gamma(\Gamma/H) &= 1 && , \text{ if } H = K; \\ \text{ch}_K^\Gamma(\Gamma/H) &= 0 && , \text{ if } K \text{ is not subconjugated to } H \text{ in } \Gamma. \end{aligned}$$

This implies

$$\text{ch}_{H_1}^\Gamma \left( \sum_{i=1}^n r_i \cdot [\Gamma/H_i] \right) = r_1$$

and hence  $r_1 = 0$ . Hence the global  $L^2$ -character map is injective. If  $\Gamma$  has only finitely many conjugacy classes of finite subgroups, then the source and target of  $\text{ch}^\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$  are rational vector spaces of the same finite dimension and  $\text{ch}^\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$  must be bijective. ■

**Remark 8.14** Suppose that there are only finitely many conjugacy classes  $(H_1), (H_2), \dots, (H_r)$  of finite subgroups in  $\Gamma$ . Without loss of generality we can assume that  $H_i$  subconjugated to  $H_j$  implies  $i \geq j$ . With respect to the obvious ordered basis for the source and target the map  $\text{ch}^\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$  is described by an upper triangular matrix  $A$  with ones on the diagonal. One can get an explicit inverse  $A^{-1}$  which again has ones on the diagonal. This leads to a characterization of the image of  $A(\Gamma)$  under the global  $L^2$ -character map  $\chi^\Gamma$ . Namely, an element in  $\eta \in \prod_{i=1}^r \mathbb{Q}$  lies in  $\text{ch}^\Gamma(A(\Gamma))$  if and only if the following *Burnside integrality conditions* are satisfied

$$A^{-1}\eta \in \prod_{i=1}^r \mathbb{Z}. \quad (8.15)$$

Now suppose that  $\Gamma$  is finite. Then the global  $L^2$ -character map is related to the classical character map by the factor  $|WK|^{-1}$ , i.e. we have for each subgroup  $K$  of  $\Gamma$  and any finite  $\Gamma$ -set  $S$

$$\text{ch}_K^\Gamma(S) = |WK|^{-1} \cdot |S^K|. \quad (8.16)$$

One easily checks that under the identification 8.16 the integrality conditions 8.15 correspond to the classical Burnside ring congruences for finite groups [13, section 5.8], [14, section IV.5].

■

Let  $E(\Gamma, \mathcal{FIN})$  be the classifying  $\Gamma$ -space for the family  $\mathcal{FIN}$  of finite subgroups. This  $\Gamma$ -CW-complex is characterized up to  $\Gamma$ -homotopy by the property that its  $H$ -fixed point set is contractible if  $H \subset \Gamma$  is finite and empty otherwise. It is also called the classifying space for proper  $\Gamma$ -spaces and denoted by  $\underline{E}\Gamma$  in the literature. For more information about  $E(\Gamma, \mathcal{FIN})$  we refer for instance to [3], [12, section 7], [14, section I.6],

**Lemma 8.17** *Suppose that there is model for  $E(\Gamma, \mathcal{FIN})$  which is a finite  $\Gamma$ -CW-complex. Then there are only finitely many conjugacy classes of finite subgroups and for a finite subgroup  $K \subset \Gamma$*

$$\text{ch}_K^\Gamma(\chi^\Gamma(E(\Gamma, \mathcal{FIN}))) = \chi^{(2)}(WK).$$

*If  $\Gamma$  is amenable, then we get for a finite subgroup  $K \subset \Gamma$*

$$\text{ch}_K^\Gamma(\chi^\Gamma(E(\Gamma, \mathcal{FIN}))) = |WK|^{-1},$$

*where  $|WK|^{-1}$  is to be understood as 0 for infinite  $WK$ .*

**Proof** : We get from Lemma 8.11 since  $E(\Gamma, \mathcal{FIN})^K$  is a model for  $E(WK, \mathcal{FIN})$

$$\text{ch}_K^\Gamma(\chi^\Gamma(E(\Gamma, \mathcal{FIN}))) = \chi^{(2)}(E(WK, \mathcal{FIN}); \mathcal{N}(WK)).$$

Now apply Theorem 8.4.3. and Corollary 5.13. ■

**Example 8.18** Let  $1 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow \mathbb{Z}/p \longrightarrow 1$  be an extension of groups for  $n \geq 1$  and a prime number  $p$ . Then  $E(\Gamma, \mathcal{FIN})$  can be chosen as a finite  $\Gamma$ -CW-complex because only



the following cases can occur. If  $\Gamma$  contains a finite subgroup, then  $\Gamma$  is a semi-direct product of  $\mathbb{Z}^n$  and  $\mathbb{Z}/l$  and one can construct a finite  $\Gamma$ -CW-complex as model for  $E(\Gamma, \mathcal{FIN})$  with  $\mathbb{R}^n$  as underlying space. If the group  $\Gamma$  contains no finite subgroup, then  $\Gamma$  is an extension of finitely generated free abelian groups and hence  $B\Gamma$  can be chosen as a finite CW-complex. We want to compute  $\chi^\Gamma(E(\Gamma, \mathcal{FIN}))$ . The conjugation action of  $\Gamma$  on the normal subgroup  $\mathbb{Z}^n$  factorizes through the projection  $\Gamma \longrightarrow \mathbb{Z}/p$  into a operation  $\rho$  of  $\mathbb{Z}/p$  onto  $\mathbb{Z}^n$ . If this operation has a non-trivial fixed point, then  $WH$  is infinite for any finite subgroup  $H$  of  $\Gamma$  and we conclude from Lemma 8.13 and Theorem 8.17 that

$$\chi^\Gamma(E(\Gamma, \mathcal{FIN})) = 0.$$

Now suppose that this operation  $\rho$  has no non-trivial fixed points. Let  $H_0$  be the trivial subgroup and  $H_1, H_2, \dots, H_r$  be a complete set of representatives of the conjugacy classes of finite subgroups. Each  $H_i$  is isomorphic to  $\mathbb{Z}/l$ . One easily checks that there is a bijection

$$H^1(\mathbb{Z}/p; \mathbb{Z}_\rho^n) \longrightarrow \{(H) \mid H \subset \Gamma, 1 < |H| < \infty\},$$

if  $\mathbb{Z}_\rho^n$  denotes the  $\mathbb{Z}[\mathbb{Z}/l]$ -module given by  $\mathbb{Z}^n$  and  $\rho$ . We compute using 8.12

$$\begin{aligned} \text{ch}_{H_0}^\Gamma(\Gamma/H_0) &= 1; \\ \text{ch}_{H_0}^\Gamma(\Gamma/H_j) &= \frac{1}{p} & j = 1, 2, \dots, r; \\ \text{ch}_{H_i}^\Gamma(\Gamma/H_j) &= 1 & i = j, i, j = 1, 2, \dots, r; \\ \text{ch}_{H_i}^\Gamma(\Gamma/H_j) &= 0 & i \neq j, i, j = 1, 2, \dots, r; . \end{aligned}$$

We conclude

$$\chi^\Gamma(E(\Gamma, \mathcal{FIN})) = -\frac{r}{p} \cdot [\Gamma/H_0] + \sum_{i=1}^r [\Gamma/H_i].$$

The integrality conditions of 8.15 become in this case

$$\begin{aligned} \eta_0 - \frac{1}{p} \cdot \sum_{i=1}^r \eta_i &\in \mathbb{Z}; \\ \eta_i &\in \mathbb{Z} \quad i = 1, 2, \dots, r. \quad \blacksquare \end{aligned}$$

## 9. Values of $L^2$ -Betti numbers

In this section we investigate the possible values of  $L^2$ -Betti numbers.

**Conjecture 9.1** *Let  $\Gamma$  be a group and let  $X$  be a free finite  $\Gamma$ -CW-complex. Then*

$$b_p^{(2)}(X; \mathcal{N}(\Gamma)) \in \mathbb{Q}.$$

*If  $d$  is a positive integer such that the order of any finite subgroup of  $\Gamma$  divides  $d$ , then*

$$d \cdot b_p^{(2)}(X; \mathcal{N}(\Gamma)) \in \mathbb{Z}. \quad \blacksquare$$

The significance of Conjecture 9.1 and its relation to a question of Atiyah [2, page 72] about the rationality of the analytic  $L^2$ -Betti numbers of 0.1 are explained in [35, Section 2]. Let  $\mathcal{C}$  be the smallest class of groups which contains all free groups, is closed under directed unions and satisfies  $G \in \mathcal{C}$  whenever  $G$  contains a normal subgroup  $H$  such that  $H$  belongs to  $\mathcal{C}$  and  $G/H$  is elementary amenable. Conjecture 9.1 has been proven for groups  $\Gamma \in \mathcal{C}$  by Linnell [26], provided that there is an upper bound on the order of finite subgroups of  $\Gamma$ .

**Theorem 9.2** *1. Let  $\Gamma$  be a group such that there is no bound on the order of finite subgroups. Then*

*(a) Given  $\beta \in [0, \infty]$ , there is a countably generated projective  $\mathbb{Z}\Gamma$ -module  $P$  satisfying*

$$\dim_{\mathcal{N}(\Gamma)}(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} P) = \beta;$$

*(b) Given a sequence  $\beta_3, \beta_4, \dots$  of elements in  $[0, \infty]$ , there is a free  $\Gamma$ -CW-complex  $X$  satisfying*

$$b_p^{(2)}(X; \mathcal{N}(\Gamma)) = \beta_p \quad \text{for } p \geq 3;$$

*If  $\Gamma$  is countably presented, one can arrange that  $X$  has countably many  $\Gamma$ -equivariant cells;*

*2. Let  $\Gamma$  be a group such that there is a bound on the order of finite subgroups. Let  $d$  be the least common multiple of the orders of finite subgroups of  $\Gamma$ . Suppose that Conjecture 9.1 holds for  $\Gamma$ . Then we get for any  $\Gamma$ -space  $X$  and  $p \geq 0$*

$$d \cdot b_p^{(2)}(X; \mathcal{N}(\Gamma)) \in \mathbb{Z} \cup \{\infty\};$$

*3. Given a sequence of elements  $\beta_1, \beta_2, \dots$  in  $[0, \infty]$ , there is a countable group  $\Gamma$  with  $b_p^{(2)}(\Gamma) = \beta_p$  for  $p \geq 1$ . If  $\beta_1$  is rational,  $\Gamma$  can be chosen to be finitely generated.*

**Proof:** 1.a.) Since there is no bound on the order of finite subgroups, we can find a sequence of finite subgroups  $H_1, H_2, \dots$  of  $\Gamma$  such that  $\beta = \sum_{i=1}^{\infty} |H_i|^{-1}$ . Then

$$P = \oplus_{i=1}^{\infty} \mathbb{C}[\Gamma/H_i]$$

is the desired module by Theorem 3.4.1 and Additivity and Cofinality (see Theorem 0.6.4.).

1.b.) By assertion 1.a) we can choose a sequence of countably generated projective  $\mathbb{C}\Gamma$ -modules  $P_3, P_4, \dots$  such that for  $p \geq 3$

$$\dim_{\mathcal{N}(\Gamma)}(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} P_p) = \beta_p. \quad (9.3)$$

Next we construct inductively a nested sequence  $X_2 \subset X_3 \subset \dots$  of  $\Gamma$ -CW-complexes together with  $\Gamma$ -retractions  $r_p : X_p \longrightarrow X_{p-1}$  for  $p \geq 3$  such that  $X_2$  is the 2-skeleton of a model for  $E\Gamma$ ,  $X_p$  is obtained from  $X_{p-1}$  by attaching countably many free  $\Gamma$ -equivariant  $p$ -cells and  $p+1$ -cells and

$$H_n(X_p, X_{p-1}) = \begin{cases} P_p & n = p \\ 0 & n \neq p \end{cases}. \quad (9.4)$$

The Eilenberg-swindle yields a split-exact sequence  $0 \longrightarrow C_{p+1} \xrightarrow{c_{p+1}} C_p \longrightarrow P_p \longrightarrow 0$  of  $\mathbb{C}\Gamma$ -modules such that  $C_{p+1}$  and  $C_p$  are countably generated free  $\mathbb{Z}\Gamma$ -modules with a basis. Now one attaches for each element of the basis of  $C_p$  trivially a free  $\Gamma$ -equivariant  $p$ -cell to  $X$ . Then one attaches for each element of the basis of  $C_{p+1}$  a free  $\Gamma$ -equivariant  $p+1$ -cell to  $X$ , where the attaching maps are chosen such that the cellular  $\mathbb{C}\Gamma$ -chain complex of  $(Y, X)$  is just the  $\mathbb{Z}\Gamma$ -chain complex which is concentrated in dimension  $p+1$  and  $p$  and given there by  $C_{p+1} \xrightarrow{c_{p+1}} C_p$ . Details of the construction of the  $\Gamma$ -CW-complexes  $X_p$  and  $\Gamma$ -retractions  $r_p$  can be found in [37, Theorem 2.2, page 201], [45]. Now define  $Y = \operatorname{colim}_{p \rightarrow \infty} X_p$ . One easily checks for  $p \geq 3$

$$\begin{aligned} H_p^\Gamma(Y; \mathcal{N}(\Gamma)) &= \mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} P_p; \\ b_p^{(2)}(Y; \mathcal{N}(\Gamma)) &= \beta_p. \end{aligned}$$

2.) Let  $f : Y \longrightarrow X$  be a  $\Gamma$ -CW-approximation of  $X$  [30, page 35], i.e. a  $\Gamma$ -CW-complex  $Y$  with a  $\Gamma$ -map  $f$  such that  $f^H$  is a weak homotopy equivalence and hence a weak homology equivalence [47, Theorem IV.7.15 on page 182] for  $H \subset \Gamma$ . Then  $b_p^{(2)}(Y; \mathcal{N}(\Gamma)) = b_p^{(2)}(X; \mathcal{N}(\Gamma))$  for all  $p \geq 0$  by Lemma 4.8.2. Hence we can assume without loss of generality that  $X$  is a  $\Gamma$ -CW-complex.

Conjecture 9.1 is equivalent to the statement that for any finitely presented  $\mathbb{Z}\Gamma$ -module  $M$

$$d \cdot \dim_{\mathcal{N}(\Gamma)}(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} M) \in \mathbb{Z}. \quad (9.5)$$

This follows essentially from [35, Lemma 2.2]. Now let  $D_*$  be any  $\mathbb{Z}\Gamma$ -chain complex such that  $D_p$  is isomorphic to  $\bigoplus_{i=1}^r \mathbb{Z}[\Gamma/H_i]$  for some non-negative integer  $r$  and finite subgroups  $H_i$ . Then the cokernel of each of the differentials  $d_p$  is a finitely presented  $\mathbb{Z}\Gamma$ -module and 9.5 yields for all  $p \geq 0$  using Additivity (see Theorem 0.6.4) and Lemma 3.4.1

$$\begin{aligned} d \cdot \dim_{\mathcal{N}(\Gamma)}(\operatorname{cok}(\operatorname{id}_{\mathcal{N}(\Gamma)} \otimes_{\mathbb{Z}\Gamma} d_p)) &\in \mathbb{Z}; \\ d \cdot \dim_{\mathcal{N}(\Gamma)}(\operatorname{im}(\operatorname{id}_{\mathcal{N}(\Gamma)} \otimes_{\mathbb{Z}\Gamma} d_p)) &\in \mathbb{Z}; \\ d \cdot \dim_{\mathcal{N}(\Gamma)}(\operatorname{ker}(\operatorname{id}_{\mathcal{N}(\Gamma)} \otimes_{\mathbb{Z}\Gamma} d_p)) &\in \mathbb{Z}; \\ d \cdot \dim_{\mathcal{N}(\Gamma)}(H_p(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} D_*)) &\in \mathbb{Z}. \end{aligned} \quad (9.6)$$

Let  $X^\infty$  be the  $\Gamma$ -CW-subcomplex of  $X$  consisting of points whose isotropy groups are infinite. The sequence  $0 \longrightarrow C_*(X^\infty) \longrightarrow C_*(X) \longrightarrow C_*(X, X^\infty) \longrightarrow 0$  of cellular  $\mathbb{Z}\Gamma$ -chain complexes is exact. Since it is  $\mathbb{Z}\Gamma$ -split exact in each dimension, the sequence obtained by tensoring with  $\mathcal{N}(\Gamma)$  is still exact. The associated long exact homology sequence, Additivity (see Theorem 0.6.4) and Lemma 3.4.1 imply for  $p \geq 0$

$$\dim_{\mathcal{N}(\Gamma)}(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} C_p(X^\infty)) = 0; \quad (9.7)$$

$$b_p^{(2)}(X; \mathcal{N}(\Gamma)) = \dim_{\mathcal{N}(\Gamma)}(H_p(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} C_*(X, X^\infty))). \quad (9.8)$$

Notice that  $C_p(X, X^\infty)$  is a sum of  $\mathbb{Z}\Gamma$ -modules of the shape  $\mathbb{Z}[\Gamma/H]$  for finite groups  $H \subset \Gamma$ . Hence  $C_*(X, X^\infty)$  is a colimit over a directed set  $I$  of subcomplexes  $D_*[i]$  (directed by inclusion) such that each  $D_p[i]$  is isomorphic to  $\bigoplus_{i=1}^r \mathbb{Z}[\Gamma/H_i]$  for some non-negative integer  $r$  and finite subgroups  $H_i$ . Since homology commutes with colimits we conclude from 9.8 and Theorem 2.9.2

$$\begin{aligned} b_p^{(2)}(X; \mathcal{N}(\Gamma)) &= \sup\{\inf\{\dim_{\mathcal{N}(\Gamma)}(\operatorname{im}(H_p(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} D_*[i]) \longrightarrow H_p(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} D_*[j])) \\ &\quad \mid j \in I, i \leq j\} \mid i \in I\}. \end{aligned}$$

Since the set  $\{r \in \mathbb{R} \mid d \cdot r \in \mathbb{Z}\}$  is discrete in  $\mathbb{R}$ , it suffices to show for each inclusion  $\iota : D_*[i] \longrightarrow D_*[j]$  and all  $p \geq 0$

$$d \cdot \dim_{\mathcal{N}(\Gamma)} (\text{im}(\iota_* : H_p(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} D_*[i]) \longrightarrow H_p(\mathcal{N}(\Gamma) \otimes_{\mathbb{Z}\Gamma} D_*[j]))) \in \mathbb{Z}. \quad (9.9)$$

Let  $F_*$  be any acyclic  $\mathcal{N}(\Gamma)$ -chain complex with  $F_p = 0$  for  $p < 0$  such that  $d \cdot \dim(F_p) \in \mathbb{Z}$  holds for all  $p \geq 0$ . Then we get  $d \cdot \dim(\text{im}(f_p : F_p \longrightarrow F_{p-1})) \in \mathbb{Z}$  for all  $p \geq 0$  since we have the short exact sequences  $0 \longrightarrow \text{im}(f_{p+1}) \longrightarrow F_p \longrightarrow \text{im}(f_p) \longrightarrow 0$  and  $\text{im}(f_1) = F_0$ . Hence we obtain 9.9 from 9.6 and the conclusion above for the case where  $F$  is the long exact homology sequence of the pair  $(\text{cyl}(\iota), D_*[i])$  since there is a  $\mathbb{Z}\Gamma$ -chain homotopy equivalence from the mapping cylinder  $\text{cyl}(\iota)$  to  $D_*[j]$  whose composition with the inclusion of  $D_*[i]$  in  $\text{cyl}(\iota)$  is  $\iota$ .

3.) is proven in [11, section 4]. This finishes the proof of Theorem 9.2. ■

**Remark 9.10** The group  $\Gamma = \prod_{i=1}^{\infty} \mathbb{Z} * \mathbb{Z}$  satisfies  $H_p^{\Gamma}(E\Gamma; \mathcal{N}(\Gamma)) = 0$  for all  $p \geq 0$ . This is interesting in connection with the zero-in-the-spectrum conjecture ([28], [35, section 11]).

■

## References

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Version of February 7, 2008